The Schulze Method of Voting

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Summary. In recent years, the Pirate Party of Sweden, the Wikimedia Foundation, the Debian project, the Gentoo project, and many other private organizations adopted a new single-winner election method for internal elections and referendums. In this paper, we will introduce this method, demonstrate that it satisfies e.g. resolvability, Condorcet, Schwartz, Smith-IIA, Pareto, reversal symmetry, monotonicity, prudence, and independence of clones and present an O(C^3) algorithm to calculate the winner, where C is the number of alternatives.

Keywords and Phrases: Condorcet criterion, independence of clones, monotonicity, Pareto efficiency, reversal symmetry, single-winner election methods, prudent ranking rules

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Contents

Symbols

$\wedge$	... and ...
$\vee$	... or ...
$\forall$	... for all ...
$\exists$	... there is at least one ...
$\in$	... element of ...
$\notin$	... not element of ...
$\Rightarrow$	... then ...
$\Leftrightarrow$	... then and only then ...
$\mathbb{N}$	natural numbers without zero, $\mathbb{N} = \{1, 2, 3, ...\}$
$\mathbb{N}_0$	natural numbers with zero, $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$
$\mathbb{R}$	real numbers
$\emptyset$	the empty set

1. Introduction

One important property of a good single-winner election method is that it minimizes the number of "overruled" voters (according to some heuristic). Because of this reason, the Simpson-Kramer method, that always chooses that alternative whose worst pairwise defeat is the weakest, was very popular over a long time. However, in recent years, the Simpson-Kramer method has been criticized by many social choice theorists. Smith (1973) criticizes that this method doesn't choose from the top-set of alternatives. Tideman (1987) complains that this method is vulnerable to the strategic nomination of a large number of similar alternatives, so-called *clones*. And Saari (1994) rejects this method for violating *reversal symmetry*. A violation of reversal symmetry can lead to strange situations where still the same alternative is chosen when all ballots are reversed, meaning that the same alternative is identified as best one and simultaneously as worst one.

In this paper, we will show that only a slight modification (section 4.8) of the Simpson-Kramer method is needed so that the resulting method satisfies the criteria proposed by Smith (section 4.7), Tideman (section 4.6), and Saari (section 4.4). The resulting method will be called *Schulze method*. Random simulations by Wright (2009) confirmed that, in almost 99% of all instances, the Schulze method conforms with the Simpson-Kramer method (table 11.1). In this paper, we will prove that, nevertheless, the Schulze method still satisfies all important criteria that are also satisfied by the Simpson-Kramer method, like resolvability (section 4.2), Pareto (section 4.3), monotonicity (section 4.5), and prudence (section 4.9). Because of these reasons, already several private organizations have adopted the Schulze method.

1997 – 2006: In 1997, I proposed the Schulze method to a large number of people, who are interested in mathematical aspects of election methods. This method was discussed for the first time in a public mailing list between June 1998 and November 1998 (e.g. Ossipoff, 1998; Petry, 1998; Schulze, 1998), when it was discussed at the *Election-Methods mailing list*. In June 2003, the Debian project, a software developer organization with about 1,000 eligible members, adopted this method in a referendum with 144 against 16 votes; Debian GNU/Linux is the largest and most popular non-commercial Linux distribution. In May 2005, the Gentoo Foundation, a software developer organization with about 100 eligible members, adopted this method; Gentoo Linux is another wide-spread Linux distribution.

2007 – 2011: In 2008, 2009, and 2011, the Wikimedia Foundation, a non-profit charitable organization with about 43,000 eligible members (in 2011), used the proposed method for the election of its Board of Trustees; the Wikimedia Foundation is the umbrella organization e.g. for Wikipedia, Wiktionary, Wikiquote, Wikidata, Wikibooks, Wikisource, Wikinews, Wikivoyage, Wikiversity, and Wikispecies; it is, therefore, the fifth most important Internet corporation (after Alphabet/Google/YouTube, Facebook/WhatsApp, Yahoo!, and Baidu). In June 2008, the “Free Software Foundation Europe” (FSFE), a software project with about 1,500 eligible members, adopted this method. In July 2008, Ubuntu, a software developer organization with about 700 eligible members, adopted this method. In August 2008, “K Desktop Environment” (KDE), a software developer organization with about 200 eligible members, adopted this method. In October 2009, the “Pirate Party of Sweden” (about 3,000 eligible members) adopted this method. In May 2010, the “Pirate Party of Germany” (about 11,000 eligible members) adopted this method. In November 2010, OpenStack, a software project with about 3,000 eligible members, adopted this method. Since February 2011, the “Pirate Party of Austria” (about 300 eligible members) uses this method. Since November 2011, the “Pirate Party of Australia” (about 1,300 eligible members) uses this method.

2012 – 2017: Since January 2013, the “Pirate Party of Iceland” (about 4,000 eligible members) uses this method. Since April 2013, the associated student government at Northwestern University (about 20,000 eligible members) uses this method. Since October 2013, the “German Association of Pediatricians” (“Berufsverband der Kinder- und Jugendärzte”; BVKJ; about 12,000 eligible members) uses this method. Since October 2013, the “Five Star Movement” (“Movimento 5 Stelle”, M5S), a political party in Italy with about 140,000 eligible members, uses this method. Since May 2014, the associated student government at Albert Ludwig University of Freiburg (about 25,000 eligible members) uses this method. Since January 2015, the “Pirate Party of the Netherlands” (about 1,400 eligible members) uses this method. In February 2016, the city of Silla (about 19,000 inhabitants) in Spain adopted the Schulze method for referendums (www01 – www05). In July 2016, the “European Students’ Forum” (“Association des états généraux des étudiants de l’Europe”, AEGEE), a student organization with about 13,000 eligible members, adopted this method. Since January 2017, Podemos, a political party in Spain with about 500,000 eligible members, uses this method. In March 2017, the “Internet Corporation for Assigned Names and Numbers” (ICANN) adopted the Schulze method for the election of its board and the board of the “Address Supporting Organization” (ASO), a supporting organization affiliated with ICANN.

Today (December 2017), the proposed method is used by more than 60 organizations with more than 700,000 eligible members in total. Therefore, the proposed method is more wide-spread than all other Condorcet-consistent single-winner election methods combined. Hill (2008) even claims that MTV uses this method to decide which music videos go into rotation.

Furthermore, the proposed method is used by many Internet decision support systems, like the “Condorcet Internet Voting Service” (CIVS), GoogleVotes (Hardt and Lopes, 2015), LiquidFeedback (Behrens, 2014), Selectricity (Hill, 2008), Airesis, preftools, OpenAgora, and OpenSTV.

There has been some debate about an appropriate name for this method. Some people suggested names like “beatpath(s)”, “beatpath method”, “beatpath winner”, “beatpath matrix”, “beatpath tournament matrix”, “beatpath power ranking” (BeatPower), “path method”, “path voting”, “path winner”, “path matrix”, “Schwartz sequential dropping” (SSD), “cloneproof Schwartz sequential dropping” (CSSD), and “MinMax decision function”. Brearley (1999) suggested names like “descending minimum gross score” (DminGS), “descending minimum augmented gross score” (DminAGS), and “descending minimum doubly augmented gross score” (DminDAGS), depending on how the strength of a pairwise link is measured. Heitzig (2001) suggested names like “strong immunity from binary arguments” (SImA) and “sequential dropping towards a spanning tree” (SDST). However, I prefer the name “Schulze method”, not because of academic arrogance, but because the other names do not refer to the method itself but to specific heuristics for implementing it, and so may mislead readers into believing that no other method for implementing it is possible.

In section 2 of this paper, the Schulze method is defined. In section 3, this method is applied to concrete examples. In section 4, this method is analyzed. Detailed descriptions of this method can also be found in publications by Schulze (2003, 2011), Tideman (2006, pages 228–232), Stahl and Johnson (2006, 2017), McCaffrey (2008a, 2008b), Börgers (2009, pages 37–42), Camps (2012a, 2012b, 2013, 2014a, 2014b, 2014c), Behrens (2014), D. Müller (2014, 2015), Moses (2017), and Pattinson (2017). This method is also described and discussed in papers by Green-Armytage (2004), Taylor (2004), Meskanen and Nurmi (2006a, 2006b, 2008), Yue (2007), Nebel (2009), Wright (2009), Rivest and Shen (2010), Abisheva (2012), Bucovetsky (2012), Gaspers (2012), Grünheid (2012, 2015, 2016), Negriu (2012), Parkes and Xia (2012), Happes (2013), Menton (2013a, 2013b), J. Müller (2013), Parkes and Seuken (2013), Felsenthal and Tideman (2014), Li (2014), Mattei (2014), Reisch (2014), Schend (2015), Baumeister and Rothe (2016), Bubboloni and Gori (2016), Caragiannis (2016), Contucci (2016), Darlington (2016), Diethelm (2016), Fischer (2016), Hemaspaandra (2016), Pan (2016), Ruiz-Padillo (2016), Shah (2016), Aziz (2017), Becirovic (2017), Hazra (2017), Hoang (2017), Izetta (2017), Louridas (2017), Pérez-Fernández (2017a, 2017b), Sekar (2017), Skowron (2017), and Tozer (2017). Applications of the Schulze method are described in papers by Narizzano (2006a, 2006b, 2006c, 2007), Ghersi (2007), Callison-Burch (2009), Arguello (2011a, 2011b, 2011c, 2017), Audhkhasi (2011), Gelder (2011), Maheswari (2012), Muldoon (2012), Oryńczak (2012), Prati (2012), Bohne (2013, 2015), Zhou (2013, 2014), Akbib (2014a, 2014b), Garg (2014), Lawonn (2014), Pallett (2014), Wang (2014), Baer (2015), Bountris (2015), Degeest (2015), Evita (2015), Nguyen (2015), Plösch (2015), Proag (2015), Aswatha (2016), Cai (2016), Chen (2016), Mangeli (2016), Vargas (2016), Verdiesen (2016), Xexéo (2016), Goel (2017), Işıklı (2017), Moal (2017), and Rijnsburger (2017). Cases, where the Schulze method is used to evaluate empirical data, are mentioned by Morales (2008), Wimmer (2009, 2010), Kowalski (2013), Casadebaig (2014), Vaughan (2016), Gervits (2017), and Al-Rousan (2017).

2. Definition of the Schulze Method

2.1. Preliminaries

We presume that A is a finite and non-empty set of alternatives. $C \in \mathbb{N}$ with $1 < C < \infty$ is the number of alternatives in A.

A binary relation $\succ$ on A is *asymmetric* if it has the following property:

$\forall\ a,b \in A$, exactly one of the following three statements is valid:

1. $a \succ b$.
2. $b \succ a$.
3. $a \approx b$ (where “$a \approx b$” means “neither $a \succ b$ nor $b \succ a$”).

A binary relation $\succ$ on A is *irreflexive* if it has the following property:

$\forall\ a \in A$: $a \approx a$.

A binary relation $\succ$ on A is *transitive* if it has the following property:

$\forall\ a,b,c \in A$: (($a \succ b$ and $b \succ c$) $\Rightarrow$ $a \succ c$).

A binary relation $\succ$ on A is *negatively transitive* if it has the following property (where “$a \succsim b$” means “not $b \succ a$”):

$\forall\ a,b,c \in A$: (($a \succsim b$ and $b \succsim c$) $\Rightarrow$ $a \succsim c$).

A binary relation $\succ$ on A is *linear* (or *total* or *complete*) if it has the following property:

$\forall\ a,b \in A$: ($b \in A \setminus \{a\}$ $\Rightarrow$ ($a \succ b$ or $b \succ a$)).

A *strict partial order* is an asymmetric, irreflexive, and transitive relation. A *strict weak order* is a strict partial order that is also negatively transitive. A *linear order* (or *total order* or *complete order*) is a strict weak order that is also linear. A *profile* is a finite and non-empty list of strict weak orders each on A.

Input of the proposed method is a profile V. $N \in \mathbb{N}$ with $0 < N < \infty$ is the number of strict weak orders in $V := \{ \succ_1, ..., \succ_N \}$. These strict weak orders will sometimes be called “voters” or “ballots”.

Suppose $V_1 := \{ \succ_1, ..., \succ_{N_1} \}$ and $V_2 := \{ \succ_{1'}, ..., \succ_{N_2'} \}$ are two profiles each on the same set of alternatives A. Then the concatenation of these two profiles will be denoted $V_1 + V_2 := \{ \succ_1, ..., \succ_{N_1}, \succ_{1'}, ..., \succ_{N_2'} \}$.

“$a \succ_v b$” means “voter $v \in V$ strictly prefers alternative $a \in A$ to alternative b”. “$a \approx_v b$” means “voter $v \in V$ is indifferent between alternative a and alternative b”. “$a \succsim_v b$” means “$a \succ_v b$ or $a \approx_v b$”.

Output of the proposed method is (1) a strict partial order $\mathcal{O}$ on A and (2) a set $\emptyset \neq \mathcal{S} \subseteq A$ of potential winners.

A possible implementation of the Schulze method looks as follows:

> Each voter gets a complete list of all alternatives and ranks these alternatives in order of preference. The individual voter may give the same preference to more than one alternative and he may keep alternatives unranked. When a given voter does not rank all alternatives, then this means (1) that this voter strictly prefers all ranked alternatives to all not ranked alternatives and (2) that this voter is indifferent between all not ranked alternatives. The individual voter may also skip preferences; however, skipping preferences has no impact on the result of the elections since only the cast order of the preferences matters, not the absolute numbers.

Suppose $N[e,f] := \| \{ v \in V \mid e \succ_v f \} \|$ is the number of voters who strictly prefer alternative e to alternative f. We presume that the strength of the link ef depends only on $N[e,f]$ and $N[f,e]$. Therefore, the strength of the link ef can be denoted $(N[e,f],N[f,e])$. We presume that a binary relation $\succ_D$ on $\mathbb{N}_0 \times \mathbb{N}_0$ is defined such that the link ef is stronger than the link gh if and only if $(N[e,f],N[f,e]) \succ_D (N[g,h],N[h,g])$. $N[e,f]$ is the *support* for the link ef; $N[f,e]$ is its *opposition*.

Example 1 (*margin*):

When the strength of the link *ef* is measured by *margin*, then its strength is the difference $N[e,f] - N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{margin} (N[g,h],N[h,g])$ if and only if $N[e,f] - N[f,e] > N[g,h] - N[h,g]$.

Example 2 (*ratio*):

When the strength of the link *ef* is measured by *ratio*, then its strength is the ratio $N[e,f] / N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{ratio} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] \cdot N[h,g] > N[f,e] \cdot N[g,h]$.
4. $N[e,f] > N[g,h]$ and $N[f,e] \leq N[h,g]$.
5. $N[e,f] \geq N[g,h]$ and $N[f,e] < N[h,g]$.

Example 3 (*winning votes*):

When the strength of the link *ef* is measured by *winning votes*, then its strength is measured primarily by its support $N[e,f]$.

$(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.
5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] < N[h,g]$.
6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

Example 4 (*losing votes*):

When the strength of the link *ef* is measured by *losing votes*, then its strength is measured primarily by its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{los} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] < N[h,g]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.
5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] > N[g,h]$.
6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.

The most intuitive definitions for the strength of a link are its *margin* and its *ratio*. However, we only presume that $\succ_D$ is a strict weak order on $\mathbb{N}_0 \times \mathbb{N}_0$.

For some proofs, we have to make additional presumptions for $\succ_D$. We will state explicitly when and where we take use of additional presumptions. Typical additional presumptions for $\succ_D$ are:

(2.1.1) (*positive responsiveness*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0:$$
$$(\ (\ x_1 > y_1 \wedge x_2 \leq y_2\ ) \vee (\ x_1 \geq y_1 \wedge x_2 < y_2\ )\ ) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2).$$

(2.1.2) (*reversal symmetry*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0:$$
$$(x_1,x_2) \succ_D (y_1,y_2) \Rightarrow (y_2,y_1) \succ_D (x_2,x_1).$$

(2.1.3) (*homogeneity*)

$$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0\ \forall\ c_1,c_2 \in \mathbb{N}:$$
$$(c_1 \cdot x_1,c_1 \cdot x_2) \succ_D (c_1 \cdot y_1,c_1 \cdot y_2) \Rightarrow (c_2 \cdot x_1,c_2 \cdot x_2) \succ_D (c_2 \cdot y_1,c_2 \cdot y_2).$$

The presumption, that the strength of the link ef depends only on $N[e,f]$ and $N[f,e]$, guarantees (1) that the proposed method satisfies anonymity and neutrality, (2) that adding a ballot, on which all alternatives are ranked equally, cannot change the result of the elections, and (3) that the proposed method is a C2 *Condorcet social choice function* (CSCF) according to Fishburn’s (1977) terminology.

Presumption (2.1.1) says that, when the support of a link increases and its opposition doesn’t increase or when its opposition decreases and its support doesn’t decrease, then the strength of this link increases. So presumption (2.1.1) says that the strength of a link responds to a change of its support or its opposition in the correct manner. Presumption (2.1.1) guarantees that the proposed method satisfies resolvability (section 4.2), Pareto (section 4.3), and monotonicity (section 4.5). When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy presumption (2.1.1), are identical.

Presumption (2.1.2) says that, the stronger the link (x_1,x_2) gets, the weaker the opposite link (x_2,x_1) gets. Presumption (2.1.2) basically says that, when the individual ballots $\succ_v$ are reversed for all voters $v \in V$, then also the order of the links $(x_1,x_2) \succ_D (y_1,y_2)$ is reversed.

Homogeneity means that the result depends only on the proportion of ballots of each type, not on their absolute numbers. Presumption (2.1.3) guarantees that the proposed method satisfies homogeneity.

$\succ_{margin}$, $\succ_{ratio}$, $\succ_{win}$, and $\succ_{los}$ each satisfy (2.1.1) – (2.1.3).

Corollary (2.1.4):

If $\succ_D$ satisfies presumption (2.1.2), then all ties have equivalent strengths. In short:

(2.1.4) $\forall\ x,y \in \mathbb{N}_0: (x,x) \approx_D (y,y).$

Proof of corollary (2.1.4):

Suppose $(x,x) \succ_D (y,y)$ for some $x,y \in \mathbb{N}_0$. Then with (2.1.2), we get $(y,y) \succ_D (x,x)$. But this is a contradiction to the presumption $(x,x) \succ_D (y,y)$ and to the presumption that $\succ_D$ is a strict weak order. □

Corollary (2.1.5):

If $\succ_D$ satisfies presumptions (2.1.1) and (2.1.2), then (i) every pairwise victory is stronger than every pairwise tie and (ii) every pairwise tie is stronger than every pairwise defeat. In short:

(2.1.5) (*majority*)

$\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0:$
$(\ (\ x_1 > x_2 \wedge y_1 \leq y_2\ ) \vee (\ x_1 \geq x_2 \wedge y_1 < y_2\ )\ ) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2).$

Proof of corollary (2.1.5):

Suppose $(x_1,x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $x_1 > x_2$ is a victory.

Suppose $(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $y_1 = y_2$ is a tie.

Suppose $(z_1,z_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $z_1 < z_2$ is a defeat.

With (2.1.1), we get: $(x_1,x_2) \succ_D (x_2,x_2)$.

With (2.1.4), we get: $(x_2,x_2) \approx_D (y_1,y_2)$.

With (2.1.4), we get: $(y_1,y_2) \approx_D (z_1,z_1)$.

With (2.1.1), we get: $(z_1,z_1) \succ_D (z_1,z_2)$.

Therefore, we get: $(x_1,x_2) \succ_D (x_2,x_2) \approx_D (y_1,y_2) \approx_D (z_1,z_1) \succ_D (z_1,z_2)$.

Thus, we get (2.1.5). □

Suppose $\emptyset \neq \mathcal{M} \subset \mathbb{N}_0 \times \mathbb{N}_0$ is finite and non-empty. Then “$\max_D\mathcal{M}$”, the *set of maximum elements* of $\mathcal{M}$, and “$\min_D\mathcal{M}$”, the *set of minimum elements* of $\mathcal{M}$, are defined as follows: $(\beta_1,\beta_2) \in \max_D\mathcal{M}$ if and only if (1) $(\beta_1,\beta_2) \in \mathcal{M}$ and (2) $(\beta_1,\beta_2) \succsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$. $(\gamma_1,\gamma_2) \in \min_D\mathcal{M}$ if and only if (1) $(\gamma_1,\gamma_2) \in \mathcal{M}$ and (2) $(\gamma_1,\gamma_2) \precsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$.

We write “$(\beta_1,\beta_2) := \max_D\mathcal{M}$” and “$(\gamma_1,\gamma_2) := \min_D\mathcal{M}$” for “$(\beta_1,\beta_2)$ is an arbitrarily chosen element of $\max_D\mathcal{M}$” and “(γ_1,γ_2) is an arbitrarily chosen element of $\min_D\mathcal{M}$”.

2.2. Basic Definitions

In this section, the Schulze method is defined. Concrete examples can be found in section 3.

Basic idea of the Schulze method is that the *strength* of the indirect comparison "alternative a vs. alternative b" is the *strength* of the *strongest path* $a \equiv c(1),...,c(n) \equiv b$ from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$ and that the *strength* of a path is the *strength* $(N[c(i),c(i+1)],N[c(i+1),c(i)])$ of its *weakest link* $c(i),c(i+1)$.

The Schulze method is defined as follows:

> A *path* from alternative $x \in A$ to alternative $y \in A \setminus \{x\}$ is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:
>
> 1. $x \equiv c(1)$.
> 2. $y \equiv c(n)$.
> 3. $n \in \mathbb{N}$ with $2 \leq n < \infty$.
> 4. For all $i = 1,...,(n–1)$: $c(i+1) \in A \setminus \{c(i)\}$.
>
> The *strength* of the path $c(1),...,c(n)$ is
> $$\min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n–1) \}.$$
>
> In other words: The strength of a path is the strength of its weakest link.
>
> When a path $c(1),...,c(n)$ has the strength $(z_1,z_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, then the *critical links* of this path are the links with $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \approx_D (z_1,z_2)$.
>
> $$P_D[a,b] := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n–1) \} \mid c(1),...,c(n) \text{ is a path from alternative } a \text{ to alternative } b \}.$$
>
> In other words: $P_D[a,b] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$.
>
> (2.2.1) The binary relation $\mathcal{O}$ on A is defined as follows:
> $$ab \in \mathcal{O} :\Leftrightarrow P_D[a,b] \succ_D P_D[b,a].$$
>
> (2.2.2) $\mathcal{S} := \{ a \in A \mid \forall\, b \in A \setminus \{a\}: ba \notin \mathcal{O} \}$ is the *set of potential winners*.

When there is only one potential winner $\mathcal{S} = \{a\}$, then this alternative is a *unique winner*.

When $P_D[a,b] \succ_D P_D[b,a]$, then we say "alternative a disqualifies alternative b" or "alternative a dominates alternative b".

As the link ab is already a path from alternative a to alternative b of strength $(N[a,b],N[b,a])$, we get

(2.2.3) $\forall\ a,b \in A$: $P_D[a,b] \succsim_D (N[a,b],N[b,a])$.

With (2.2.1) and (2.2.3), we get

(2.2.4) $(N[a,b],N[b,a]) \succ_D P_D[b,a] \Rightarrow ab \in \mathcal{O}$.

Furthermore, we get

(2.2.5) $\forall\ a,b,c \in A$: $\min_D \{\ P_D[a,b], P_D[b,c]\ \} \precsim_D P_D[a,c]$.

Otherwise, if $\min_D \{\ P_D[a,b], P_D[b,c]\ \}$ was strictly larger than $P_D[a,c]$, then this would be a contradiction to the definition of $P_D[a,c]$ since there would be a path from alternative a to alternative c via alternative b with a strength of more than $P_D[a,c]$.

Furthermore, we get

(2.2.6) $\forall\ a,b \in A$: $P_D[a,b] \precsim_D \max_D \{\ (N[a,c],N[c,a]) \mid c \in A \setminus \{a\}\ \}$.

(2.2.7) $\forall\ a,b \in A$: $P_D[a,b] \precsim_D \max_D \{\ (N[c,b],N[b,c]) \mid c \in A \setminus \{b\}\ \}$.

The asymmetry of $\mathcal{O}$ follows directly from the asymmetry of $\succ_D$. The irreflexivity of $\mathcal{O}$ follows directly from the irreflexivity of $\succ_D$. Furthermore, in section 4.1, we will see that the binary relation $\mathcal{O}$ is transitive. This guarantees that there is always at least one potential winner.

Suppose $\emptyset \neq B \subsetneq A$. Then we get

(2.2.8) $\forall\ a \in B\ \forall\ b \notin B$: $P_D[a,b] \precsim_D \max_D \{\ (N[c,d],N[d,c]) \mid c \in B \text{ and } d \notin B\ \}$.

2.3. Implementation

2.3.1. Part 1

In section 2.3.1, we explain how to calculate (1) the strict partial order $\mathcal{O}$ on A and (2) the set $\varnothing \neq \mathcal{S} \subseteq A$ of potential winners, as defined in section 2.2.

The strength $P_D[i,j]$ of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$ can be calculated with the Floyd-Warshall (Floyd, 1962; Warshall, 1962) algorithm. The runtime to calculate the strengths of all strongest paths is O(C^3), where C is the number of alternatives in A.

Input: $N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

Output: $P_D[i,j] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$pred[i,j] \in A \setminus \{j\}$ is the predecessor of alternative j in the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$\mathcal{O}$ is the binary relation as defined in (2.2.1).

“*winner*[i] = *true*” if and only if $i \in \mathcal{S}$.

Stage 1 (initialization):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            P_D[i,j] : = (N[i,j],N[j,i])
            pred[i,j] : = i
        end
    end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            for k : = 1 to C
            begin
                if ( i ≠ k ) then
                begin
                    if ( j ≠ k ) then
                    begin
                        if ( P_D[j,k] ≺_D min_D { P_D[j,i], P_D[i,k] } ) then
                        begin
                            P_D[j,k] : = min_D { P_D[j,i], P_D[i,k] }
                            if ( pred[j,k] ≠ pred[i,k] ) then
                            begin
                                pred[j,k] : = pred[i,k]
                            end
                        end
                    end
                end
            end
        end
    end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the set of potential winners):

```
for i : = 1 to C
begin
    winner[i] : = true
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            if ( P_D[j,i] ≻_D P_D[i,j] ) then
            begin
                ji ∈ O
                winner[i] : = false
            end
            else
            begin
                ji ∉ O
            end
        end
    end
end
```

(α) It cannot be stressed frequently enough that the order of the indices in the triple-loop of the Floyd-Warshall algorithm is *not* irrelevant. When i is the index of the outer loop of the triple-loop of the Floyd-Warshall algorithm, then the clause (line 24) must be “ if ($P_D[j,k] \prec_D \min_D \{ P_D[j,i], P_D[i,k] \}$) ”. Otherwise, it is not guaranteed that a single pass through the triple-loop of the Floyd-Warshall algorithm is sufficient to find the strongest paths.

(β) With the predecessor matrix $pred[i,j]$, we can recursively determine the strongest paths. Suppose we want to determine the strongest path $c(1),...,c(n)$ from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$. Then we start with

$$n := 1$$

$$d(1) := b$$

We repeat

$$n := n + 1$$

$$d(n) := pred[a,d(n-1)]$$

until we get $d(n) = a$ for some $n \in \mathbb{N}$. The strongest path $c(1),...,c(n)$ from alternative a to alternative b is then given by $d(n),...,d(1)$.

(γ) The runtime to calculate the pairwise matrix is $O(N \cdot (C\hat{}2))$. The runtime of the Floyd-Warshall algorithm, as defined in this section, is $O(C\hat{}3)$. Therefore, the total runtime to calculate the binary relation $\mathcal{O}$, as defined in (2.2.1), and the set $\mathcal{S}$, as defined in (2.2.2), is $O(N \cdot (C\hat{}2) + C\hat{}3)$.

2.3.2. Part 2

In section 2.3.2, we explain how to check whether a concrete alternative $m \in A$ is a potential winner.

Sometimes, we don't want to calculate all potential winners. We only want to check for a concrete alternative *m* whether it is a potential winner. In this case, we don't have to calculate the strengths $P_D[i,j]$ of the strongest paths from every alternative $i \in A$ to every other alternative $j \in A \setminus \{i\}$. It is sufficient to calculate the strengths of the strongest paths from alternative *m* to every other alternative $i \in A \setminus \{m\}$ and the strengths of the strongest paths from every other alternative $i \in A \setminus \{m\}$ to alternative *m*. This can be done with the Dijkstra (1959) algorithm in a runtime O(*C*^2).

Input: $N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$m \in A$ is that alternative for which we want to check whether it is a potential winner.

Output: $P_D[m,i] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative *m* to alternative $i \in A \setminus \{m\}$.

$P_D[i,m] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $i \in A \setminus \{m\}$ to alternative *m*.

“*winner* = *true*” if and only if *m* is a potential winner.

Stage 1 (initialization):

```
n : = 1
if ( m = 1 ) then
begin
    n : = 2
end
```

Stage 2 (calculation of the strengths of the strongest paths from alternative *m* to every other alternative $i \in A \setminus \{m\}$):

```
for i : = 1 to C
begin
    if ( i ≠ m ) then
    begin
        P_D[m,i] : = (N[m,i],N[i,m])
        marked[i] : = false
    end
end
marked[m] : = true
for i : = 1 to ( C – 1 )
begin
    (x_1,x_2) : = P_D[m,n]
    j : = n
    for k : = 1 to C
    begin
        if ( marked[k] = false ) then
        begin
            if  ( ( (x_1,x_2) ≺_D P_D[m,k] ) or ( marked[j] = true ) ) then
            begin
                (x_1,x_2) : = P_D[m,k]
                j : = k
            end
        end
    end
    marked[j] : = true
    for k : = 1 to C
    begin
        if ( marked[k] = false ) then
        begin
            if  ( P_D[m,k] ≺_D min_D { P_D[m,j], (N[j,k],N[k,j]) } ) then
            begin
                P_D[m,k] : = min_D { P_D[m,j], (N[j,k],N[k,j]) }
            end
        end
    end
end
```

Stage 3 (calculation of the strengths of the strongest paths from every other alternative $i \in A \setminus \{m\}$ to alternative m):

```
for i : = 1 to C
begin
    if ( i ≠ m ) then
    begin
        P_D[i,m] : = (N[i,m],N[m,i])
        marked[i] : = false
    end
end
marked[m] : = true
for i : = 1 to ( C – 1 )
begin
    (x_1,x_2) : = P_D[n,m]
    j : = n
    for k : = 1 to C
    begin
        if ( marked[k] = false ) then
        begin
            if ( ( (x_1,x_2) ≺_D P_D[k,m] ) or ( marked[j] = true ) ) then
            begin
                (x_1,x_2) : = P_D[k,m]
                j : = k
            end
        end
    end
    marked[j] : = true
    for k : = 1 to C
    begin
        if ( marked[k] = false ) then
        begin
            if ( P_D[k,m] ≺_D min_D { P_D[j,m], (N[k,j],N[j,k]) } ) then
            begin
                P_D[k,m] : = min_D { P_D[j,m], (N[k,j],N[j,k]) }
            end
        end
    end
end
```

Stage 4 (checking whether alternative m is a potential winner):

```
winner : = true
for i : = 1 to C
begin
    if ( i ≠ m ) then
    begin
        if ( P_D[i,m] ≻_D P_D[m,i] ) then
        begin
            winner : = false
        end
    end
end
```

2.3.3. Part 3

Suppose that we have already guessed or determined that the statement “ $ab \in \mathcal{O}$ ” is true. In section 2.3.3, we will show how we can then demonstrate the correctness of this statement.

To demonstrate that the statement “ $ab \in \mathcal{O}$ ” is true, we have to present a $(x_1,x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that (1) there is a path from alternative a to alternative b with a strength of at least (x_1,x_2) and (2) there is no path from alternative b to alternative a with a strength of at least (x_1,x_2).

To demonstrate that there is a path from alternative a to alternative b with a strength of at least (x_1,x_2), we can simply use the sequence $c(1),...,c(n)$ as calculated in remark β of section 2.3.1 or the path as determined in section 2.3.2 or a path found by guesswork. The runtime to verify that a given sequence is really a path from alternative a to alternative b with a strength of at least (x_1,x_2) is O(C).

When there is no path from alternative b to alternative a with a strength of at least (x_1,x_2), we can demonstrate this by presenting two sets B_1 and B_2 such that

(2.3.3.1) $b \in B_1$.

(2.3.3.2) $a \in B_2$.

(2.3.3.3) $B_1 \cup B_2 = A$.

(2.3.3.4) $B_1 \cap B_2 = \emptyset$.

(2.3.3.5) $\forall\ i \in B_1\ \forall\ j \in B_2$: $(N[i,j],N[j,i]) \prec_D (x_1,x_2)$.

When B_1 and B_2 are given, then the runtime to verify that (2.3.3.1) – (2.3.3.5) are satisfied is O(C^2).

(α) Suppose that we have <u>*not*</u> calculated the strengths of the strongest paths from every alternative $i \in A$ to every other alternative $j \in A \setminus \{i\}$, but that we have found a path from alternative a to alternative b of strength $(x_1,x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ and want to check whether this path is sufficient so that alternative a disqualifies alternative b (i.e. $ab \in \mathcal{O}$).

Then we can calculate the sets B_1 and B_2, for example, with the “breadth-first search” (BFS) algorithm as follows. The runtime to calculate the sets B_1 and B_2 is O(C^2).

Input: $N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$(x_1,x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$a,b \in A$ are those alternatives for which we want to show that there is no path from alternative b to alternative a with a strength of at least (x_1,x_2).

Output: the sets B_1 and B_2 as described above

```
B1 : = {b}
m : = 1
array1 [1] : = b
while ( m > 0 ) do
begin
    n : = m
    for k : = 1 to m
    begin
        array2 [k] : = array1 [k]
    end
    m : = 0
    for i : = 1 to n
    begin
        j : = array2 [i]
        for k : = 1 to C
        begin
            if ( k ∉ B1 ) then
            begin
                if ( (N[j,k],N[k,j]) ≈D (x1,x2) ) then
                begin
                    B1 : = B1 ∪ {k}
                    m : = m + 1
                    array1 [m] : = k
                end
            end
        end
    end
end
B2 : = A \ B1
```

When, at some point, alternative a is added to the set B_1, then this means that a path from alternative a to alternative b of strength (x_1,x_2) is <u>*not*</u> sufficient so that alternative a disqualifies alternative b.

(β) Suppose (1) that we have calculated the strengths of the strongest paths from every alternative $i \in A$ to every other alternative $j \in A \setminus \{i\}$, as described in section 2.3.1, and (2) that the statement " $ab \in \mathcal{O}$ " is true. Then B_1 and B_2 are given as follows:

$$B_1 := (\{b\} \cup \{ c \in A \mid P_D[b,c] \gtrsim_D (x_1,x_2) \}).$$

$$B_2 := A \setminus B_1.$$

3. Examples

Throughout section 3, we presume that $\succ_D$ satisfies (2.1.1) so that, when each voter $v \in V$ casts a linear order $\succ_v$ on A, all definitions for $\succ_D$ are identical.

3.1. Example 1

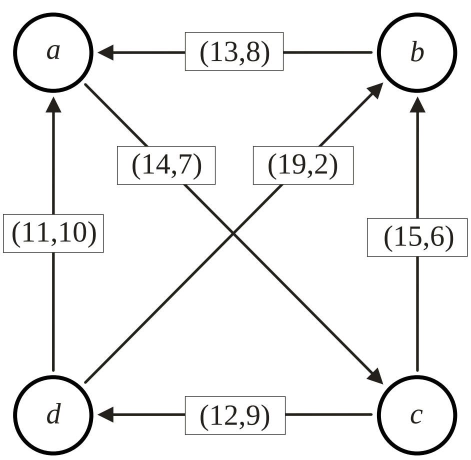
Example 1:

8 voters	$a \succ_v c \succ_v d \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
4 voters	$c \succ_v d \succ_v b \succ_v a$
4 voters	$d \succ_v b \succ_v a \succ_v c$
3 voters	$d \succ_v c \succ_v b \succ_v a$

$N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$. In example 1, the pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	8	14	10
$N[b,*]$	13	---	6	2
$N[c,*]$	7	15	---	12
$N[d,*]$	11	19	9	---

The following digraph illustrates the graph theoretic interpretation of pairwise elections. If $N[i,j] > N[j,i]$, then there is a link from vertex i to vertex j of strength $(N[i,j],N[j,i])$:



The above digraph can be used to determine the strengths of the strongest paths. In the following, “x, (Z_1,Z_2), y” means “$(N[x,y],N[y,x]) = (Z_1,Z_2)$”.

$a \rightarrow b$: There are 2 paths from alternative a to alternative b.

Path 1: a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (14,7), (15,6) } $\approx_D$ (14,7).

Path 2: a, (14,7), c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (14,7), (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative a to alternative b is $\max_D$ { (14,7), (12,9) } $\approx_D$ (14,7).

$a \rightarrow c$: There is only one path from alternative a to alternative c.

Path 1: a, (14,7), c with a strength of (14,7).

$a \rightarrow d$: There is only one path from alternative a to alternative d.

Path 1: a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (14,7), (12,9) } $\approx_D$ (12,9).

$b \rightarrow a$: There is only one path from alternative b to alternative a.

Path 1: b, (13,8), a with a strength of (13,8).

$b \rightarrow c$: There is only one path from alternative b to alternative c.

Path 1: b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (13,8), (14,7) } $\approx_D$ (13,8).

$b \rightarrow d$: There is only one path from alternative b to alternative d.

Path 1: b, (13,8), a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (13,8), (14,7), (12,9) } $\approx_D$ (12,9).

$c \rightarrow a$: There are 3 paths from alternative c to alternative a.

Path 1: c, (15,6), b, (13,8), a
with a strength of $\min_D$ { (15,6), (13,8) } $\approx_D$ (13,8).

Path 2: c, (12,9), d, (11,10), a
with a strength of $\min_D$ { (12,9), (11,10) } $\approx_D$ (11,10).

Path 3: c, (12,9), d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (12,9), (19,2), (13,8) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative a is $\max_D$ { (13,8), (11,10), (12,9) } $\approx_D$ (13,8).

$c \to b$: There are 2 paths from alternative c to alternative b.

Path 1: c, (15,6), b with a strength of (15,6).

Path 2: c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative b is $\max_D$ { (15,6), (12,9) } $\approx_D$ (15,6).

$c \to d$: There is only one path from alternative c to alternative d.

Path 1: c, (12,9), d with a strength of (12,9).

$d \to a$: There are 2 paths from alternative d to alternative a.

Path 1: d, (11,10), a with a strength of (11,10).

Path 2: d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (19,2), (13,8) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative a is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

$d \to b$: There are 2 paths from alternative d to alternative b.

Path 1: d, (11,10), a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (11,10), (14,7), (15,6) } $\approx_D$ (11,10).

Path 2: d, (19,2), b with a strength of (19,2).

So the strength of the strongest path from alternative d to alternative b is $\max_D$ { (11,10), (19,2) } $\approx_D$ (19,2).

$d \to c$: There are 2 paths from alternative d to alternative c.

Path 1: d, (11,10), a, (14,7), c
with a strength of $\min_D$ { (11,10), (14,7) } $\approx_D$ (11,10).

Path 2: d, (19,2), b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (19,2), (13,8), (14,7) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative c is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

The following table lists the strongest paths. The critical links of the strongest paths are underlined:

	... to a	... to b	... to c	... to d
from a ...	---	a, $\underline{(14,7)}$, c, (15,6), b	a, $\underline{(14,7)}$, c	a, (14,7), c, $\underline{(12,9)}$, d
from b ...	b, $\underline{(13,8)}$, a	---	b, $\underline{(13,8)}$, a, (14,7), c	b, (13,8), a, (14,7), c, $\underline{(12,9)}$, d
from c ...	c, (15,6), b, $\underline{(13,8)}$, a	c, $\underline{(15,6)}$, b	---	c, $\underline{(12,9)}$, d
from d ...	d, (19,2), b, $\underline{(13,8)}$, a	d, $\underline{(19,2)}$, b	d, (19,2), b, $\underline{(13,8)}$, a, (14,7), c	---

The strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(14,7)	(14,7)	(12,9)
$P_D[b,*]$	(13,8)	---	(13,8)	(12,9)
$P_D[c,*]$	(13,8)	(15,6)	---	(12,9)
$P_D[d,*]$	(13,8)	(19,2)	(13,8)	---

$xy \in \mathcal{O}$ if and only if $P_D[x,y] \succ_D P_D[y,x]$. So in example 1, we get $\mathcal{O} = \{ab, ac, cb, da, db, dc\}$.

$x \in \mathcal{S}$ if and only if $yx \notin \mathcal{O}$ for all $y \in A \setminus \{x\}$. So in example 1, we get $\mathcal{S} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C-1) \cdot (C-2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(6,15)	(13,8)	(14,7)	b	a	$P_D[b,c]$ is updated from (6,15) to (13,8); $pred[b,c]$ is updated from b to a
2	a	b	d	(2,19)	(13,8)	(10,11)	b	a	$P_D[b,d]$ is updated from (2,19) to (10,11); $pred[b,d]$ is updated from b to a
3	a	c	b	(15,6)	(7,14)	(8,13)	c	a	
4	a	c	d	(12,9)	(7,14)	(10,11)	c	a	
5	a	d	b	(19,2)	(11,10)	(8,13)	d	a	
6	a	d	c	(9,12)	(11,10)	(14,7)	d	a	$P_D[d,c]$ is updated from (9,12) to (11,10); $pred[d,c]$ is updated from d to a
7	b	a	c	(14,7)	(8,13)	(13,8)	a	a	
8	b	a	d	(10,11)	(8,13)	(10,11)	a	a	
9	b	c	a	(7,14)	(15,6)	(13,8)	c	b	$P_D[c,a]$ is updated from (7,14) to (13,8); $pred[c,a]$ is updated from c to b
10	b	c	d	(12,9)	(15,6)	(10,11)	c	a	
11	b	d	a	(11,10)	(19,2)	(13,8)	d	b	$P_D[d,a]$ is updated from (11,10) to (13,8); $pred[d,a]$ is updated from d to b
12	b	d	c	(11,10)	(19,2)	(13,8)	a	a	$P_D[d,c]$ is updated from (11,10) to (13,8)
13	c	a	b	(8,13)	(14,7)	(15,6)	a	c	$P_D[a,b]$ is updated from (8,13) to (14,7); $pred[a,b]$ is updated from a to c
14	c	a	d	(10,11)	(14,7)	(12,9)	a	c	$P_D[a,d]$ is updated from (10,11) to (12,9); $pred[a,d]$ is updated from a to c
15	c	b	a	(13,8)	(13,8)	(13,8)	b	b	
16	c	b	d	(10,11)	(13,8)	(12,9)	a	c	$P_D[b,d]$ is updated from (10,11) to (12,9); $pred[b,d]$ is updated from a to c
17	c	d	a	(13,8)	(13,8)	(13,8)	b	b	
18	c	d	b	(19,2)	(13,8)	(15,6)	d	c	
19	d	a	b	(14,7)	(12,9)	(19,2)	c	d	
20	d	a	c	(14,7)	(12,9)	(13,8)	a	a	
21	d	b	a	(13,8)	(12,9)	(13,8)	b	b	
22	d	b	c	(13,8)	(12,9)	(13,8)	a	a	
23	d	c	a	(13,8)	(12,9)	(13,8)	b	b	
24	d	c	b	(15,6)	(12,9)	(19,2)	c	d	

3.2. Example 2

The following example is by Hoag and Hallett (1926, page 502), where the authors use this example to illustrate their proposal (*Hallett count*).

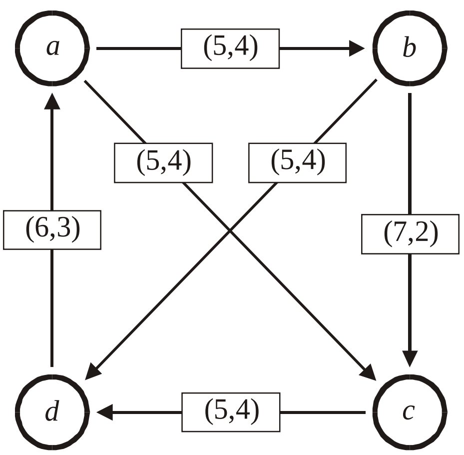
Example 2:

3 voters	$a \succ_v b \succ_v c \succ_v d$
2 voters	$c \succ_v b \succ_v d \succ_v a$
2 voters	$d \succ_v a \succ_v b \succ_v c$
2 voters	$d \succ_v b \succ_v c \succ_v a$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	5	5	3
$N[b,*]$	4	---	7	5
$N[c,*]$	4	2	---	5
$N[d,*]$	6	4	4	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (5,4), *b*	*a*, (5,4), *c*	*a*, (5,4), *b*, (5,4), *d*
from *b* ...	*b*, (5,4), *d*, (6,3), *a*	---	*b*, (7,2), *c*	*b*, (5,4), *d*
from *c* ...	*c*, (5,4), *d*, (6,3), *a*	*c*, (5,4), *d*, (6,3), *a*, (5,4), *b*	---	*c*, (5,4), *d*
from *d* ...	*d*, (6,3), *a*	*d*, (6,3), *a*, (5,4), *b*	*d*, (6,3), *a*, (5,4), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(5,4)	(5,4)	(5,4)
$P_D[b,*]$	(5,4)	---	(7,2)	(5,4)
$P_D[c,*]$	(5,4)	(5,4)	---	(5,4)
$P_D[d,*]$	(6,3)	(5,4)	(5,4)	---

We get $\mathcal{O} = \{bc, da\}$ and $\mathcal{S} = \{b, d\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(7,2)	(4,5)	(5,4)	b	a	
2	a	b	d	(5,4)	(4,5)	(3,6)	b	a	
3	a	c	b	(2,7)	(4,5)	(5,4)	c	a	$P_D[c,b]$ is updated from (2,7) to (4,5); $pred[c,b]$ is updated from c to a
4	a	c	d	(5,4)	(4,5)	(3,6)	c	a	
5	a	d	b	(4,5)	(6,3)	(5,4)	d	a	$P_D[d,b]$ is updated from (4,5) to (5,4); $pred[d,b]$ is updated from d to a
6	a	d	c	(4,5)	(6,3)	(5,4)	d	a	$P_D[d,c]$ is updated from (4,5) to (5,4); $pred[d,c]$ is updated from d to a
7	b	a	c	(5,4)	(5,4)	(7,2)	a	b	
8	b	a	d	(3,6)	(5,4)	(5,4)	a	b	$P_D[a,d]$ is updated from (3,6) to (5,4); $pred[a,d]$ is updated from a to b
9	b	c	a	(4,5)	(4,5)	(4,5)	c	b	
10	b	c	d	(5,4)	(4,5)	(5,4)	c	b	
11	b	d	a	(6,3)	(5,4)	(4,5)	d	b	
12	b	d	c	(5,4)	(5,4)	(7,2)	a	b	
13	c	a	b	(5,4)	(5,4)	(4,5)	a	a	
14	c	a	d	(5,4)	(5,4)	(5,4)	b	c	
15	c	b	a	(4,5)	(7,2)	(4,5)	b	c	
16	c	b	d	(5,4)	(7,2)	(5,4)	b	c	
17	c	d	a	(6,3)	(5,4)	(4,5)	d	c	
18	c	d	b	(5,4)	(5,4)	(4,5)	a	a	
19	d	a	b	(5,4)	(5,4)	(5,4)	a	a	
20	d	a	c	(5,4)	(5,4)	(5,4)	a	a	
21	d	b	a	(4,5)	(5,4)	(6,3)	b	d	$P_D[b,a]$ is updated from (4,5) to (5,4); $pred[b,a]$ is updated from b to d
22	d	b	c	(7,2)	(5,4)	(5,4)	b	a	
23	d	c	a	(4,5)	(5,4)	(6,3)	c	d	$P_D[c,a]$ is updated from (4,5) to (5,4); $pred[c,a]$ is updated from c to d
24	d	c	b	(4,5)	(5,4)	(5,4)	a	a	$P_D[c,b]$ is updated from (4,5) to (5,4)

3.3. Example 3

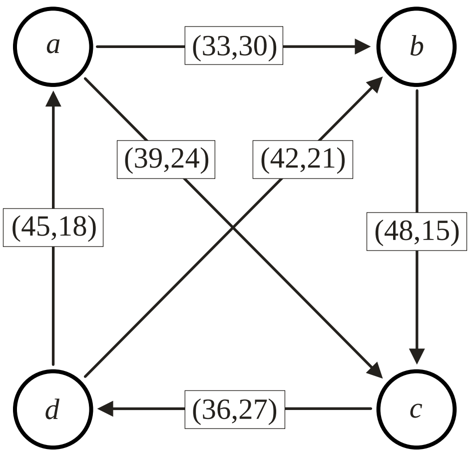
Example 3:

12 voters	$a \succ_v b \succ_v c \succ_v d$
6 voters	$a \succ_v d \succ_v b \succ_v c$
9 voters	$b \succ_v c \succ_v d \succ_v a$
15 voters	$c \succ_v d \succ_v a \succ_v b$
21 voters	$d \succ_v b \succ_v a \succ_v c$

The pairwise matrix *N* looks as follows:

	N[*,*a*]	*N*[*,*b*]	*N*[*,*c*]	*N*[*,*d*]
N[*a*,*]	---	33	39	18
N[*b*,*]	30	---	48	21
N[*c*,*]	24	15	---	36
N[*d*,*]	45	42	27	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (39,24), *c*, (36,27), *d*, (42,21), *b*	*a*, (39,24), *c*	*a*, (39,24), *c*, (36,27), *d*
from *b* ...	*b*, (48,15), *c*, (36,27), *d*, (45,18), *a*	---	*b*, (48,15), *c*	*b*, (48,15), *c*, (36,27), *d*
from *c* ...	*c*, (36,27), *d*, (45,18), *a*	*c*, (36,27), *d*, (42,21), *b*	---	*c*, (36,27), *d*
from *d* ...	*d*, (45,18), *a*	*d*, (42,21), *b*	*d*, (42,21), *b*, (48,15), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(36,27)	(39,24)	(36,27)
$P_D[b,*]$	(36,27)	---	(48,15)	(36,27)
$P_D[c,*]$	(36,27)	(36,27)	---	(36,27)
$P_D[d,*]$	(45,18)	(42,21)	(42,21)	---

We get $\mathcal{O} = \{ac, bc, da, db, dc\}$ and $\mathcal{S} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(48,15)	(30,33)	(39,24)	b	a	
2	a	b	d	(21,42)	(30,33)	(18,45)	b	a	
3	a	c	b	(15,48)	(24,39)	(33,30)	c	a	$P_D[c,b]$ is updated from (15,48) to (24,39); $pred[c,b]$ is updated from c to a
4	a	c	d	(36,27)	(24,39)	(18,45)	c	a	
5	a	d	b	(42,21)	(45,18)	(33,30)	d	a	
6	a	d	c	(27,36)	(45,18)	(39,24)	d	a	$P_D[d,c]$ is updated from (27,36) to (39,24); $pred[d,c]$ is updated from d to a
7	b	a	c	(39,24)	(33,30)	(48,15)	a	b	
8	b	a	d	(18,45)	(33,30)	(21,42)	a	b	$P_D[a,d]$ is updated from (18,45) to (21,42); $pred[a,d]$ is updated from a to b
9	b	c	a	(24,39)	(24,39)	(30,33)	c	b	
10	b	c	d	(36,27)	(24,39)	(21,42)	c	b	
11	b	d	a	(45,18)	(42,21)	(30,33)	d	b	
12	b	d	c	(39,24)	(42,21)	(48,15)	a	b	$P_D[d,c]$ is updated from (39,24) to (42,21); $pred[d,c]$ is updated from a to b
13	c	a	b	(33,30)	(39,24)	(24,39)	a	a	
14	c	a	d	(21,42)	(39,24)	(36,27)	b	c	$P_D[a,d]$ is updated from (21,42) to (36,27); $pred[a,d]$ is updated from b to c
15	c	b	a	(30,33)	(48,15)	(24,39)	b	c	
16	c	b	d	(21,42)	(48,15)	(36,27)	b	c	$P_D[b,d]$ is updated from (21,42) to (36,27); $pred[b,d]$ is updated from b to c
17	c	d	a	(45,18)	(42,21)	(24,39)	d	c	
18	c	d	b	(42,21)	(42,21)	(24,39)	d	a	
19	d	a	b	(33,30)	(36,27)	(42,21)	a	d	$P_D[a,b]$ is updated from (33,30) to (36,27); $pred[a,b]$ is updated from a to d
20	d	a	c	(39,24)	(36,27)	(42,21)	a	b	
21	d	b	a	(30,33)	(36,27)	(45,18)	b	d	$P_D[b,a]$ is updated from (30,33) to (36,27); $pred[b,a]$ is updated from b to d
22	d	b	c	(48,15)	(36,27)	(42,21)	b	b	
23	d	c	a	(24,39)	(36,27)	(45,18)	c	d	$P_D[c,a]$ is updated from (24,39) to (36,27); $pred[c,a]$ is updated from c to d
24	d	c	b	(24,39)	(36,27)	(42,21)	a	d	$P_D[c,b]$ is updated from (24,39) to (36,27); $pred[c,b]$ is updated from a to d

3.4. Example 4

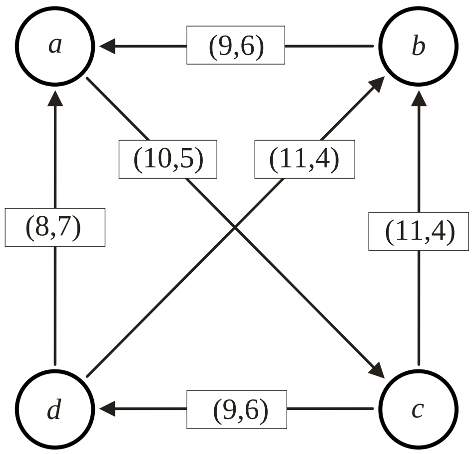
Example 4:

6	voters	$a \succ_v c \succ_v d \succ_v b$
1	voter	$b \succ_v a \succ_v d \succ_v c$
3	voters	$c \succ_v b \succ_v d \succ_v a$
3	voters	$d \succ_v b \succ_v a \succ_v c$
2	voters	$d \succ_v c \succ_v b \succ_v a$

The pairwise matrix *N* looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	6	10	7
$N[b,*]$	9	---	4	4
$N[c,*]$	5	11	---	9
$N[d,*]$	8	11	6	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (10,5), *c*, (11,4), *b*	*a*, (10,5), *c*	*a*, (10,5), *c*, (9,6), *d*
from *b* ...	*b*, (9,6), *a*	---	*b*, (9,6), *a*, (10,5), *c*	*b*, (9,6), *a*, (10,5), *c*, (9,6), *d*
from *c* ...	*c*, (11,4), *b*, (9,6), *a*	*c*, (11,4), *b*	---	*c*, (9,6), *d*
from *d* ...	*d*, (11,4), *b*, (9,6), *a*	*d*, (11,4), *b*	*d*, (11,4), *b*, (9,6), *a*, (10,5), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(10,5)	(10,5)	(9,6)
$P_D[b,*]$	(9,6)	---	(9,6)	(9,6)
$P_D[c,*]$	(9,6)	(11,4)	---	(9,6)
$P_D[d,*]$	(9,6)	(11,4)	(9,6)	---

We get $\mathcal{O} = \{ab, ac, cb, db\}$ and $\mathcal{S} = \{a, d\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(4,11)	(9,6)	(10,5)	b	a	$P_D[b,c]$ is updated from (4,11) to (9,6); $pred[b,c]$ is updated from b to a
2	a	b	d	(4,11)	(9,6)	(7,8)	b	a	$P_D[b,d]$ is updated from (4,11) to (7,8); $pred[b,d]$ is updated from b to a
3	a	c	b	(11,4)	(5,10)	(6,9)	c	a	
4	a	c	d	(9,6)	(5,10)	(7,8)	c	a	
5	a	d	b	(11,4)	(8,7)	(6,9)	d	a	
6	a	d	c	(6,9)	(8,7)	(10,5)	d	a	$P_D[d,c]$ is updated from (6,9) to (8,7); $pred[d,c]$ is updated from d to a
7	b	a	c	(10,5)	(6,9)	(9,6)	a	a	
8	b	a	d	(7,8)	(6,9)	(7,8)	a	a	
9	b	c	a	(5,10)	(11,4)	(9,6)	c	b	$P_D[c,a]$ is updated from (5,10) to (9,6); $pred[c,a]$ is updated from c to b
10	b	c	d	(9,6)	(11,4)	(7,8)	c	a	
11	b	d	a	(8,7)	(11,4)	(9,6)	d	b	$P_D[d,a]$ is updated from (8,7) to (9,6); $pred[d,a]$ is updated from d to b
12	b	d	c	(8,7)	(11,4)	(9,6)	a	a	$P_D[d,c]$ is updated from (8,7) to (9,6)
13	c	a	b	(6,9)	(10,5)	(11,4)	a	c	$P_D[a,b]$ is updated from (6,9) to (10,5); $pred[a,b]$ is updated from a to c
14	c	a	d	(7,8)	(10,5)	(9,6)	a	c	$P_D[a,d]$ is updated from (7,8) to (9,6); $pred[a,d]$ is updated from a to c
15	c	b	a	(9,6)	(9,6)	(9,6)	b	b	
16	c	b	d	(7,8)	(9,6)	(9,6)	a	c	$P_D[b,d]$ is updated from (7,8) to (9,6); $pred[b,d]$ is updated from a to c
17	c	d	a	(9,6)	(9,6)	(9,6)	b	b	
18	c	d	b	(11,4)	(9,6)	(11,4)	d	c	
19	d	a	b	(10,5)	(9,6)	(11,4)	c	d	
20	d	a	c	(10,5)	(9,6)	(9,6)	a	a	
21	d	b	a	(9,6)	(9,6)	(9,6)	b	b	
22	d	b	c	(9,6)	(9,6)	(9,6)	a	a	
23	d	c	a	(9,6)	(9,6)	(9,6)	b	b	
24	d	c	b	(11,4)	(9,6)	(11,4)	c	d	

3.5. Example 5

The basic idea for the following example has been proposed by Cretney (1998).

3.5.1. Situation #1

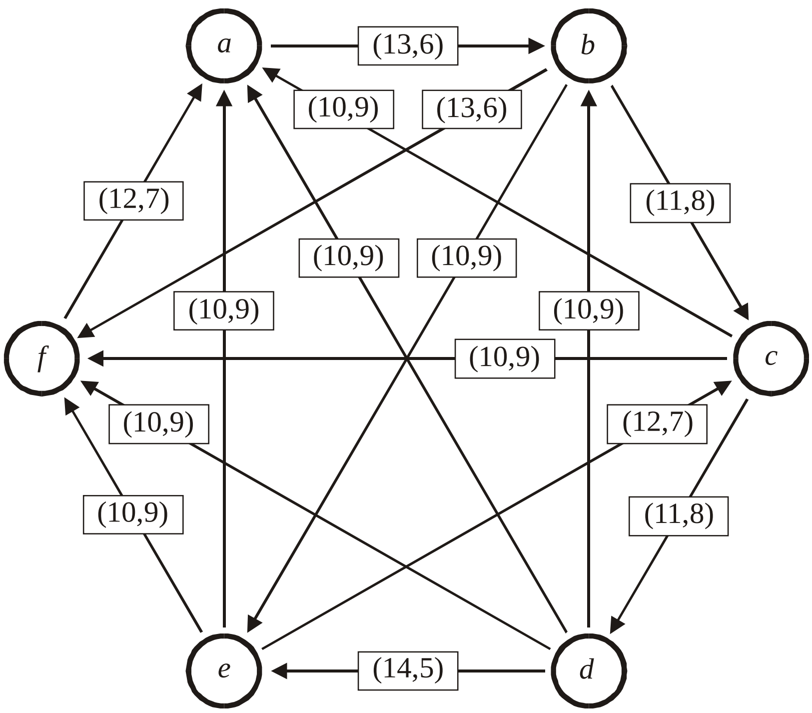
Example 5 (old):

3 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c \succ_v f$
3 voters	$b \succ_v f \succ_v e \succ_v c \succ_v d \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v f \succ_v d \succ_v e$
1 voter	$d \succ_v b \succ_v c \succ_v e \succ_v f \succ_v a$
4 voters	$d \succ_v e \succ_v f \succ_v a \succ_v b \succ_v c$
2 voters	$e \succ_v c \succ_v b \succ_v d \succ_v f \succ_v a$
2 voters	$f \succ_v a \succ_v c \succ_v d \succ_v b \succ_v e$

The pairwise matrix N^{old} looks as follows:

	$N^{old}[*,a]$	$N^{old}[*,b]$	$N^{old}[*,c]$	$N^{old}[*,d]$	$N^{old}[*,e]$	$N^{old}[*,f]$
$N^{old}[a,*]$	---	13	9	9	9	7
$N^{old}[b,*]$	6	---	11	9	10	13
$N^{old}[c,*]$	10	8	---	11	7	10
$N^{old}[d,*]$	10	10	8	---	14	10
$N^{old}[e,*]$	10	9	12	5	---	10
$N^{old}[f,*]$	12	6	9	9	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	*a*, $\underline{(13,6)}$, *b*	*a*, (13,6), *b*, $\underline{(11,8)}$, *c*	*a*, (13,6), *b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*	*a*, (13,6), *b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*, (14,5), *e*	*a*, $\underline{(13,6)}$, *b*, $\underline{(13,6)}$, *f*
from *b* ...	*b*, (13,6), *f*, $\underline{(12,7)}$, *a*	---	*b*, $\underline{(11,8)}$, *c*	*b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*	*b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*, (14,5), *e*	*b*, $\underline{(13,6)}$, *f*
from *c* ...	*c*, $\underline{(10,9)}$, *a*	*c*, $\underline{(10,9)}$, *a*, (13,6), *b*	---	*c*, $\underline{(11,8)}$, *d*	*c*, $\underline{(11,8)}$, *d*, (14,5), *e*	*c*, $\underline{(10,9)}$, *f*
from *d* ...	*d*, $\underline{(10,9)}$, *a*	*d*, $\underline{(10,9)}$, *b*	*d*, (14,5), *e*, $\underline{(12,7)}$, *c*	---	*d*, $\underline{(14,5)}$, *e*	*d*, $\underline{(10,9)}$, *f*
from *e* ...	*e*, $\underline{(10,9)}$, *a*	*e*, $\underline{(10,9)}$, *a*, (13,6), *b*	*e*, $\underline{(12,7)}$, *c*	*e*, (12,7), *c*, $\underline{(11,8)}$, *d*	---	*e*, $\underline{(10,9)}$, *f*
from *f* ...	*f*, $\underline{(12,7)}$, *a*	*f*, $\underline{(12,7)}$, *a*, (13,6), *b*	*f*, (12,7), *a*, (13,6), *b*, $\underline{(11,8)}$, *c*	*f*, (12,7), *a*, (13,6), *b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*	*f*, (12,7), *a*, (13,6), *b*, $\underline{(11,8)}$, *c*, $\underline{(11,8)}$, *d*, (14,5), *e*	---

We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, ae, af, bc, bd, be, bf, dc, de, ec, fc, fd, fe\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 120$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(11,8)	(6,13)	(9,10)	b	a	
2	a	b	d	(9,10)	(6,13)	(9,10)	b	a	
3	a	b	e	(10,9)	(6,13)	(9,10)	b	a	
4	a	b	f	(13,6)	(6,13)	(7,12)	b	a	
5	a	c	b	(8,11)	(10,9)	(13,6)	c	a	$P_D[c,b]$ is updated from (8,11) to (10,9); $pred[c,b]$ is updated from c to a
6	a	c	d	(11,8)	(10,9)	(9,10)	c	a	
7	a	c	e	(7,12)	(10,9)	(9,10)	c	a	$P_D[c,e]$ is updated from (7,12) to (9,10); $pred[c,e]$ is updated from c to a
8	a	c	f	(10,9)	(10,9)	(7,12)	c	a	
9	a	d	b	(10,9)	(10,9)	(13,6)	d	a	
10	a	d	c	(8,11)	(10,9)	(9,10)	d	a	$P_D[d,c]$ is updated from (8,11) to (9,10); $pred[d,c]$ is updated from d to a
11	a	d	e	(14,5)	(10,9)	(9,10)	d	a	
12	a	d	f	(10,9)	(10,9)	(7,12)	d	a	
13	a	e	b	(9,10)	(10,9)	(13,6)	e	a	$P_D[e,b]$ is updated from (9,10) to (10,9); $pred[e,b]$ is updated from e to a
14	a	e	c	(12,7)	(10,9)	(9,10)	e	a	
15	a	e	d	(5,14)	(10,9)	(9,10)	e	a	$P_D[e,d]$ is updated from (5,14) to (9,10); $pred[e,d]$ is updated from e to a
16	a	e	f	(10,9)	(10,9)	(7,12)	e	a	
17	a	f	b	(6,13)	(12,7)	(13,6)	f	a	$P_D[f,b]$ is updated from (6,13) to (12,7); $pred[f,b]$ is updated from f to a
18	a	f	c	(9,10)	(12,7)	(9,10)	f	a	
19	a	f	d	(9,10)	(12,7)	(9,10)	f	a	
20	a	f	e	(9,10)	(12,7)	(9,10)	f	a	
21	b	a	c	(9,10)	(13,6)	(11,8)	a	b	$P_D[a,c]$ is updated from (9,10) to (11,8); $pred[a,c]$ is updated from a to b
22	b	a	d	(9,10)	(13,6)	(9,10)	a	b	
23	b	a	e	(9,10)	(13,6)	(10,9)	a	b	$P_D[a,e]$ is updated from (9,10) to (10,9); $pred[a,e]$ is updated from a to b
24	b	a	f	(7,12)	(13,6)	(13,6)	a	b	$P_D[a,f]$ is updated from (7,12) to (13,6); $pred[a,f]$ is updated from a to b
25	b	c	a	(10,9)	(10,9)	(6,13)	c	b	
26	b	c	d	(11,8)	(10,9)	(9,10)	c	b	
27	b	c	e	(9,10)	(10,9)	(10,9)	a	b	$P_D[c,e]$ is updated from (9,10) to (10,9); $pred[c,e]$ is updated from a to b
28	b	c	f	(10,9)	(10,9)	(13,6)	c	b	
29	b	d	a	(10,9)	(10,9)	(6,13)	d	b	
30	b	d	c	(9,10)	(10,9)	(11,8)	a	b	$P_D[d,c]$ is updated from (9,10) to (10,9); $pred[d,c]$ is updated from a to b

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	b	d	e	(14,5)	(10,9)	(10,9)	d	b	
32	b	d	f	(10,9)	(10,9)	(13,6)	d	b	
33	b	e	a	(10,9)	(10,9)	(6,13)	e	b	
34	b	e	c	(12,7)	(10,9)	(11,8)	e	b	
35	b	e	d	(9,10)	(10,9)	(9,10)	a	b	
36	b	e	f	(10,9)	(10,9)	(13,6)	e	b	
37	b	f	a	(12,7)	(12,7)	(6,13)	f	b	
38	b	f	c	(9,10)	(12,7)	(11,8)	f	b	$P_D[f,c]$ is updated from (9,10) to (11,8); $pred[f,c]$ is updated from f to b
39	b	f	d	(9,10)	(12,7)	(9,10)	f	b	
40	b	f	e	(9,10)	(12,7)	(10,9)	f	b	$P_D[f,e]$ is updated from (9,10) to (10,9); $pred[f,e]$ is updated from f to b
41	c	a	b	(13,6)	(11,8)	(10,9)	a	a	
42	c	a	d	(9,10)	(11,8)	(11,8)	a	c	$P_D[a,d]$ is updated from (9,10) to (11,8); $pred[a,d]$ is updated from a to c
43	c	a	e	(10,9)	(11,8)	(10,9)	b	b	
44	c	a	f	(13,6)	(11,8)	(10,9)	b	c	
45	c	b	a	(6,13)	(11,8)	(10,9)	b	c	$P_D[b,a]$ is updated from (6,13) to (10,9); $pred[b,a]$ is updated from b to c
46	c	b	d	(9,10)	(11,8)	(11,8)	b	c	$P_D[b,d]$ is updated from (9,10) to (11,8); $pred[b,d]$ is updated from b to c
47	c	b	e	(10,9)	(11,8)	(10,9)	b	b	
48	c	b	f	(13,6)	(11,8)	(10,9)	b	c	
49	c	d	a	(10,9)	(10,9)	(10,9)	d	c	
50	c	d	b	(10,9)	(10,9)	(10,9)	d	a	
51	c	d	e	(14,5)	(10,9)	(10,9)	d	b	
52	c	d	f	(10,9)	(10,9)	(10,9)	d	c	
53	c	e	a	(10,9)	(12,7)	(10,9)	e	c	
54	c	e	b	(10,9)	(12,7)	(10,9)	a	a	
55	c	e	d	(9,10)	(12,7)	(11,8)	a	c	$P_D[e,d]$ is updated from (9,10) to (11,8); $pred[e,d]$ is updated from a to c
56	c	e	f	(10,9)	(12,7)	(10,9)	e	c	
57	c	f	a	(12,7)	(11,8)	(10,9)	f	c	
58	c	f	b	(12,7)	(11,8)	(10,9)	a	a	
59	c	f	d	(9,10)	(11,8)	(11,8)	f	c	$P_D[f,d]$ is updated from (9,10) to (11,8); $pred[f,d]$ is updated from f to c
60	c	f	e	(10,9)	(11,8)	(10,9)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
61	*d*	*a*	*b*	(13,6)	(11,8)	(10,9)	*a*	*d*	
62	*d*	*a*	*c*	(11,8)	(11,8)	(10,9)	*b*	*b*	
63	*d*	*a*	*e*	(10,9)	(11,8)	(14,5)	*b*	*d*	$P_D[a,e]$ is updated from (10,9) to (11,8); *pred*[*a*,*e*] is updated from *b* to *d*
64	*d*	*a*	*f*	(13,6)	(11,8)	(10,9)	*b*	*d*	
65	*d*	*b*	*a*	(10,9)	(11,8)	(10,9)	*c*	*d*	
66	*d*	*b*	*c*	(11,8)	(11,8)	(10,9)	*b*	*b*	
67	*d*	*b*	*e*	(10,9)	(11,8)	(14,5)	*b*	*d*	$P_D[b,e]$ is updated from (10,9) to (11,8); *pred*[*b*,*e*] is updated from *b* to *d*
68	*d*	*b*	*f*	(13,6)	(11,8)	(10,9)	*b*	*d*	
69	*d*	*c*	*a*	(10,9)	(11,8)	(10,9)	*c*	*d*	
70	*d*	*c*	*b*	(10,9)	(11,8)	(10,9)	*a*	*d*	
71	*d*	*c*	*e*	(10,9)	(11,8)	(14,5)	*b*	*d*	$P_D[c,e]$ is updated from (10,9) to (11,8); *pred*[*c*,*e*] is updated from *b* to *d*
72	*d*	*c*	*f*	(10,9)	(11,8)	(10,9)	*c*	*d*	
73	*d*	*e*	*a*	(10,9)	(11,8)	(10,9)	*e*	*d*	
74	*d*	*e*	*b*	(10,9)	(11,8)	(10,9)	*a*	*d*	
75	*d*	*e*	*c*	(12,7)	(11,8)	(10,9)	*e*	*b*	
76	*d*	*e*	*f*	(10,9)	(11,8)	(10,9)	*e*	*d*	
77	*d*	*f*	*a*	(12,7)	(11,8)	(10,9)	*f*	*d*	
78	*d*	*f*	*b*	(12,7)	(11,8)	(10,9)	*a*	*d*	
79	*d*	*f*	*c*	(11,8)	(11,8)	(10,9)	*b*	*b*	
80	*d*	*f*	*e*	(10,9)	(11,8)	(14,5)	*b*	*d*	$P_D[f,e]$ is updated from (10,9) to (11,8); *pred*[*f*,*e*] is updated from *b* to *d*
81	*e*	*a*	*b*	(13,6)	(11,8)	(10,9)	*a*	*a*	
82	*e*	*a*	*c*	(11,8)	(11,8)	(12,7)	*b*	*e*	
83	*e*	*a*	*d*	(11,8)	(11,8)	(11,8)	*c*	*c*	
84	*e*	*a*	*f*	(13,6)	(11,8)	(10,9)	*b*	*e*	
85	*e*	*b*	*a*	(10,9)	(11,8)	(10,9)	*c*	*e*	
86	*e*	*b*	*c*	(11,8)	(11,8)	(12,7)	*b*	*e*	
87	*e*	*b*	*d*	(11,8)	(11,8)	(11,8)	*c*	*c*	
88	*e*	*b*	*f*	(13,6)	(11,8)	(10,9)	*b*	*e*	
89	*e*	*c*	*a*	(10,9)	(11,8)	(10,9)	*c*	*e*	
90	*e*	*c*	*b*	(10,9)	(11,8)	(10,9)	*a*	*a*	

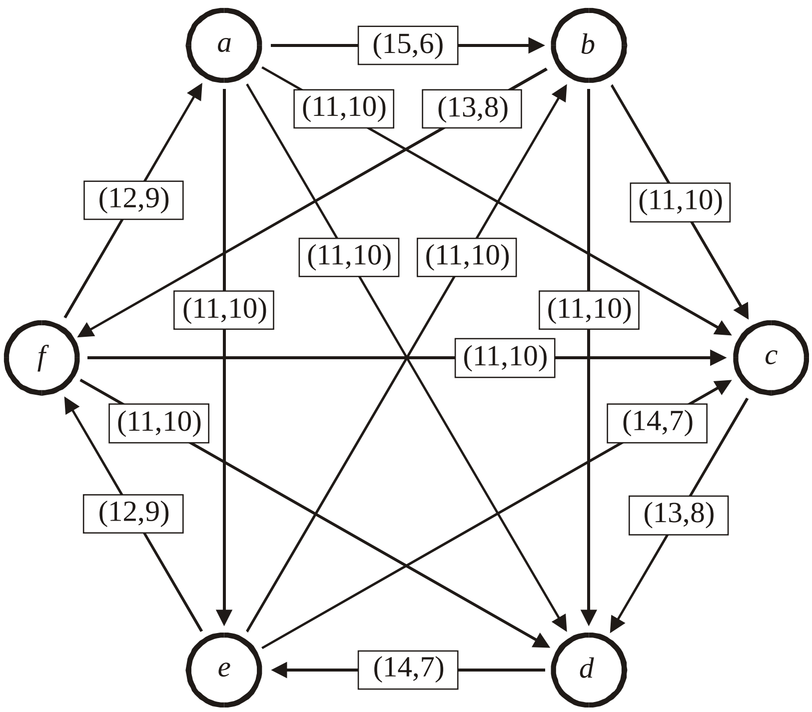
	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
91	*e*	*c*	*d*	(11,8)	(11,8)	(11,8)	*c*	*c*	
92	*e*	*c*	*f*	(10,9)	(11,8)	(10,9)	*c*	*e*	
93	*e*	*d*	*a*	(10,9)	(14,5)	(10,9)	*d*	*e*	
94	*e*	*d*	*b*	(10,9)	(14,5)	(10,9)	*d*	*a*	
95	*e*	*d*	*c*	(10,9)	(14,5)	(12,7)	*b*	*e*	$P_D[d,c]$ is updated from (10,9) to (12,7); *pred*[*d*,*c*] is updated from *b* to *e*
96	*e*	*d*	*f*	(10,9)	(14,5)	(10,9)	*d*	*e*	
97	*e*	*f*	*a*	(12,7)	(11,8)	(10,9)	*f*	*e*	
98	*e*	*f*	*b*	(12,7)	(11,8)	(10,9)	*a*	*a*	
99	*e*	*f*	*c*	(11,8)	(11,8)	(12,7)	*b*	*e*	
100	*e*	*f*	*d*	(11,8)	(11,8)	(11,8)	*c*	*c*	
101	*f*	*a*	*b*	(13,6)	(13,6)	(12,7)	*a*	*a*	
102	*f*	*a*	*c*	(11,8)	(13,6)	(11,8)	*b*	*b*	
103	*f*	*a*	*d*	(11,8)	(13,6)	(11,8)	*c*	*c*	
104	*f*	*a*	*e*	(11,8)	(13,6)	(11,8)	*d*	*d*	
105	*f*	*b*	*a*	(10,9)	(13,6)	(12,7)	*c*	*f*	$P_D[b,a]$ is updated from (10,9) to (12,7); *pred*[*b*,*a*] is updated from *c* to *f*
106	*f*	*b*	*c*	(11,8)	(13,6)	(11,8)	*b*	*b*	
107	*f*	*b*	*d*	(11,8)	(13,6)	(11,8)	*c*	*c*	
108	*f*	*b*	*e*	(11,8)	(13,6)	(11,8)	*d*	*d*	
109	*f*	*c*	*a*	(10,9)	(10,9)	(12,7)	*c*	*f*	
110	*f*	*c*	*b*	(10,9)	(10,9)	(12,7)	*a*	*a*	
111	*f*	*c*	*d*	(11,8)	(10,9)	(11,8)	*c*	*c*	
112	*f*	*c*	*e*	(11,8)	(10,9)	(11,8)	*d*	*d*	
113	*f*	*d*	*a*	(10,9)	(10,9)	(12,7)	*d*	*f*	
114	*f*	*d*	*b*	(10,9)	(10,9)	(12,7)	*d*	*a*	
115	*f*	*d*	*c*	(12,7)	(10,9)	(11,8)	*e*	*b*	
116	*f*	*d*	*e*	(14,5)	(10,9)	(11,8)	*d*	*d*	
117	*f*	*e*	*a*	(10,9)	(10,9)	(12,7)	*e*	*f*	
118	*f*	*e*	*b*	(10,9)	(10,9)	(12,7)	*a*	*a*	
119	*f*	*e*	*c*	(12,7)	(10,9)	(11,8)	*e*	*b*	
120	*f*	*e*	*d*	(11,8)	(10,9)	(11,8)	*c*	*c*	

3.5.2. Situation #2

When 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ ballots are added, then the pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$	$N^{new}[*,f]$
$N^{new}[a,*]$	---	15	11	11	11	9
$N^{new}[b,*]$	6	---	11	11	10	13
$N^{new}[c,*]$	10	10	---	13	7	10
$N^{new}[d,*]$	10	10	8	---	14	10
$N^{new}[e,*]$	10	11	14	7	---	12
$N^{new}[f,*]$	12	8	11	11	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	$a, \underline{(15,6)}, b$	$a, \underline{(11,10)}, c$	$a, \underline{(11,10)}, d$	$a, \underline{(11,10)}, e$	$a, (15,6), b, \underline{(13,8)}, f$
from *b* ...	$b, (13,8), f, \underline{(12,9)}, a$	---	$b, \underline{(11,10)}, c$	$b, \underline{(11,10)}, d$	$b, \underline{(11,10)}, d, (14,7), e$	$b, \underline{(13,8)}, f$
from *c* ...	$c, (13,8), d, (14,7), e, \underline{(12,9)}, f, \underline{(12,9)}, a$	$c, (13,8), d, (14,7), e, \underline{(12,9)}, f, \underline{(12,9)}, a, (15,6), b$	---	$c, \underline{(13,8)}, d$	$c, \underline{(13,8)}, d, (14,7), e$	$c, (13,8), d, (14,7), e, \underline{(12,9)}, f$
from *d* ...	$d, (14,7), e, \underline{(12,9)}, f, \underline{(12,9)}, a$	$d, (14,7), e, \underline{(12,9)}, f, \underline{(12,9)}, a, (15,6), b$	$d, \underline{(14,7)}, e, \underline{(14,7)}, c$	---	$d, \underline{(14,7)}, e$	$d, (14,7), e, \underline{(12,9)}, f$
from *e* ...	$e, \underline{(12,9)}, f, \underline{(12,9)}, a$	$e, \underline{(12,9)}, f, \underline{(12,9)}, a, (15,6), b$	$e, \underline{(14,7)}, c$	$e, (14,7), c, \underline{(13,8)}, d$	---	$e, \underline{(12,9)}, f$
from *f* ...	$f, \underline{(12,9)}, a$	$f, \underline{(12,9)}, a, (15,6), b$	$f, \underline{(11,10)}, c$	$f, \underline{(11,10)}, d$	$f, (12,9), a, \underline{(11,10)}, e$	---

We get $\mathcal{O}^{new} = \{ab, af, bf, ca, cb, cf, da, db, dc, de, df, ea, eb, ec, ef\}$ and $\mathcal{S}^{new} = \{d\}$.

Thus the 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ voters change the unique winner from alternative *a* to alternative *d*.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 120$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	*a*	*b*	*c*	(11,10)	(6,15)	(11,10)	*b*	*a*	
2	*a*	*b*	*d*	(11,10)	(6,15)	(11,10)	*b*	*a*	
3	*a*	*b*	*e*	(10,11)	(6,15)	(11,10)	*b*	*a*	
4	*a*	*b*	*f*	(13,8)	(6,15)	(9,12)	*b*	*a*	
5	*a*	*c*	*b*	(10,11)	(10,11)	(15,6)	*c*	*a*	
6	*a*	*c*	*d*	(13,8)	(10,11)	(11,10)	*c*	*a*	
7	*a*	*c*	*e*	(7,14)	(10,11)	(11,10)	*c*	*a*	$P_D[c,e]$ is updated from (7,14) to (10,11); *pred*[*c*,*e*] is updated from *c* to *a*
8	*a*	*c*	*f*	(10,11)	(10,11)	(9,12)	*c*	*a*	
9	*a*	*d*	*b*	(10,11)	(10,11)	(15,6)	*d*	*a*	
10	*a*	*d*	*c*	(8,13)	(10,11)	(11,10)	*d*	*a*	$P_D[d,c]$ is updated from (8,13) to (10,11); *pred*[*d*,*c*] is updated from *d* to *a*
11	*a*	*d*	*e*	(14,7)	(10,11)	(11,10)	*d*	*a*	
12	*a*	*d*	*f*	(10,11)	(10,11)	(9,12)	*d*	*a*	
13	*a*	*e*	*b*	(11,10)	(10,11)	(15,6)	*e*	*a*	
14	*a*	*e*	*c*	(14,7)	(10,11)	(11,10)	*e*	*a*	
15	*a*	*e*	*d*	(7,14)	(10,11)	(11,10)	*e*	*a*	$P_D[e,d]$ is updated from (7,14) to (10,11); *pred*[*e*,*d*] is updated from *e* to *a*
16	*a*	*e*	*f*	(12,9)	(10,11)	(9,12)	*e*	*a*	
17	*a*	*f*	*b*	(8,13)	(12,9)	(15,6)	*f*	*a*	$P_D[f,b]$ is updated from (8,13) to (12,9); *pred*[*f*,*b*] is updated from *f* to *a*
18	*a*	*f*	*c*	(11,10)	(12,9)	(11,10)	*f*	*a*	
19	*a*	*f*	*d*	(11,10)	(12,9)	(11,10)	*f*	*a*	
20	*a*	*f*	*e*	(9,12)	(12,9)	(11,10)	*f*	*a*	$P_D[f,e]$ is updated from (9,12) to (11,10); *pred*[*f*,*e*] is updated from *f* to *a*
21	*b*	*a*	*c*	(11,10)	(15,6)	(11,10)	*a*	*b*	
22	*b*	*a*	*d*	(11,10)	(15,6)	(11,10)	*a*	*b*	
23	*b*	*a*	*e*	(11,10)	(15,6)	(10,11)	*a*	*b*	
24	*b*	*a*	*f*	(9,12)	(15,6)	(13,8)	*a*	*b*	$P_D[a,f]$ is updated from (9,12) to (13,8); *pred*[*a*,*f*] is updated from *a* to *b*
25	*b*	*c*	*a*	(10,11)	(10,11)	(6,15)	*c*	*b*	
26	*b*	*c*	*d*	(13,8)	(10,11)	(11,10)	*c*	*b*	
27	*b*	*c*	*e*	(10,11)	(10,11)	(10,11)	*a*	*b*	
28	*b*	*c*	*f*	(10,11)	(10,11)	(13,8)	*c*	*b*	
29	*b*	*d*	*a*	(10,11)	(10,11)	(6,15)	*d*	*b*	
30	*b*	*d*	*c*	(10,11)	(10,11)	(11,10)	*a*	*b*	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	b	d	e	(14,7)	(10,11)	(10,11)	d	b	
32	b	d	f	(10,11)	(10,11)	(13,8)	d	b	
33	b	e	a	(10,11)	(11,10)	(6,15)	e	b	
34	b	e	c	(14,7)	(11,10)	(11,10)	e	b	
35	b	e	d	(10,11)	(11,10)	(11,10)	a	b	$P_D[e,d]$ is updated from (10,11) to (11,10); $pred[e,d]$ is updated from a to b
36	b	e	f	(12,9)	(11,10)	(13,8)	e	b	
37	b	f	a	(12,9)	(12,9)	(6,15)	f	b	
38	b	f	c	(11,10)	(12,9)	(11,10)	f	b	
39	b	f	d	(11,10)	(12,9)	(11,10)	f	b	
40	b	f	e	(11,10)	(12,9)	(10,11)	a	b	
41	c	a	b	(15,6)	(11,10)	(10,11)	a	c	
42	c	a	d	(11,10)	(11,10)	(13,8)	a	c	
43	c	a	e	(11,10)	(11,10)	(10,11)	a	a	
44	c	a	f	(13,8)	(11,10)	(10,11)	b	c	
45	c	b	a	(6,15)	(11,10)	(10,11)	b	c	$P_D[b,a]$ is updated from (6,15) to (10,11); $pred[b,a]$ is updated from b to c
46	c	b	d	(11,10)	(11,10)	(13,8)	b	c	
47	c	b	e	(10,11)	(11,10)	(10,11)	b	a	
48	c	b	f	(13,8)	(11,10)	(10,11)	b	c	
49	c	d	a	(10,11)	(10,11)	(10,11)	d	c	
50	c	d	b	(10,11)	(10,11)	(10,11)	d	c	
51	c	d	e	(14,7)	(10,11)	(10,11)	d	a	
52	c	d	f	(10,11)	(10,11)	(10,11)	d	c	
53	c	e	a	(10,11)	(14,7)	(10,11)	e	c	
54	c	e	b	(11,10)	(14,7)	(10,11)	e	c	
55	c	e	d	(11,10)	(14,7)	(13,8)	b	c	$P_D[e,d]$ is updated from (11,10) to (13,8); $pred[e,d]$ is updated from b to c
56	c	e	f	(12,9)	(14,7)	(10,11)	e	c	
57	c	f	a	(12,9)	(11,10)	(10,11)	f	c	
58	c	f	b	(12,9)	(11,10)	(10,11)	a	c	
59	c	f	d	(11,10)	(11,10)	(13,8)	f	c	
60	c	f	e	(11,10)	(11,10)	(10,11)	a	a	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
61	d	a	b	(15,6)	(11,10)	(10,11)	a	d	
62	d	a	c	(11,10)	(11,10)	(10,11)	a	a	
63	d	a	e	(11,10)	(11,10)	(14,7)	a	d	
64	d	a	f	(13,8)	(11,10)	(10,11)	b	d	
65	d	b	a	(10,11)	(11,10)	(10,11)	c	d	
66	d	b	c	(11,10)	(11,10)	(10,11)	b	a	
67	d	b	e	(10,11)	(11,10)	(14,7)	b	d	$P_D[b,e]$ is updated from (10,11) to (11,10); $pred[b,e]$ is updated from b to d
68	d	b	f	(13,8)	(11,10)	(10,11)	b	d	
69	d	c	a	(10,11)	(13,8)	(10,11)	c	d	
70	d	c	b	(10,11)	(13,8)	(10,11)	c	d	
71	d	c	e	(10,11)	(13,8)	(14,7)	a	d	$P_D[c,e]$ is updated from (10,11) to (13,8); $pred[c,e]$ is updated from a to d
72	d	c	f	(10,11)	(13,8)	(10,11)	c	d	
73	d	e	a	(10,11)	(13,8)	(10,11)	e	d	
74	d	e	b	(11,10)	(13,8)	(10,11)	e	d	
75	d	e	c	(14,7)	(13,8)	(10,11)	e	a	
76	d	e	f	(12,9)	(13,8)	(10,11)	e	d	
77	d	f	a	(12,9)	(11,10)	(10,11)	f	d	
78	d	f	b	(12,9)	(11,10)	(10,11)	a	d	
79	d	f	c	(11,10)	(11,10)	(10,11)	f	a	
80	d	f	e	(11,10)	(11,10)	(14,7)	a	d	
81	e	a	b	(15,6)	(11,10)	(11,10)	a	e	
82	e	a	c	(11,10)	(11,10)	(14,7)	a	e	
83	e	a	d	(11,10)	(11,10)	(13,8)	a	c	
84	e	a	f	(13,8)	(11,10)	(12,9)	b	e	
85	e	b	a	(10,11)	(11,10)	(10,11)	c	e	
86	e	b	c	(11,10)	(11,10)	(14,7)	b	e	
87	e	b	d	(11,10)	(11,10)	(13,8)	b	c	
88	e	b	f	(13,8)	(11,10)	(12,9)	b	e	
89	e	c	a	(10,11)	(13,8)	(10,11)	c	e	
90	e	c	b	(10,11)	(13,8)	(11,10)	c	e	$P_D[c,b]$ is updated from (10,11) to (11,10); $pred[c,b]$ is updated from c to e

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
91	e	c	d	(13,8)	(13,8)	(13,8)	c	c	
92	e	c	f	(10,11)	(13,8)	(12,9)	c	e	$P_D[c,f]$ is updated from (10,11) to (12,9); $pred[c,f]$ is updated from c to e
93	e	d	a	(10,11)	(14,7)	(10,11)	d	e	
94	e	d	b	(10,11)	(14,7)	(11,10)	d	e	$P_D[d,b]$ is updated from (10,11) to (11,10); $pred[d,b]$ is updated from d to e
95	e	d	c	(10,11)	(14,7)	(14,7)	a	e	$P_D[d,c]$ is updated from (10,11) to (14,7); $pred[d,c]$ is updated from a to e
96	e	d	f	(10,11)	(14,7)	(12,9)	d	e	$P_D[d,f]$ is updated from (10,11) to (12,9); $pred[d,f]$ is updated from d to e
97	e	f	a	(12,9)	(11,10)	(10,11)	f	e	
98	e	f	b	(12,9)	(11,10)	(11,10)	a	e	
99	e	f	c	(11,10)	(11,10)	(14,7)	f	e	
100	e	f	d	(11,10)	(11,10)	(13,8)	f	c	
101	f	a	b	(15,6)	(13,8)	(12,9)	a	a	
102	f	a	c	(11,10)	(13,8)	(11,10)	a	f	
103	f	a	d	(11,10)	(13,8)	(11,10)	a	f	
104	f	a	e	(11,10)	(13,8)	(11,10)	a	a	
105	f	b	a	(10,11)	(13,8)	(12,9)	c	f	$P_D[b,a]$ is updated from (10,11) to (12,9); $pred[b,a]$ is updated from c to f
106	f	b	c	(11,10)	(13,8)	(11,10)	b	f	
107	f	b	d	(11,10)	(13,8)	(11,10)	b	f	
108	f	b	e	(11,10)	(13,8)	(11,10)	d	a	
109	f	c	a	(10,11)	(12,9)	(12,9)	c	f	$P_D[c,a]$ is updated from (10,11) to (12,9); $pred[c,a]$ is updated from c to f
110	f	c	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[c,b]$ is updated from (11,10) to (12,9); $pred[c,b]$ is updated from e to a
111	f	c	d	(13,8)	(12,9)	(11,10)	c	f	
112	f	c	e	(13,8)	(12,9)	(11,10)	d	a	
113	f	d	a	(10,11)	(12,9)	(12,9)	d	f	$P_D[d,a]$ is updated from (10,11) to (12,9); $pred[d,a]$ is updated from d to f
114	f	d	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[d,b]$ is updated from (11,10) to (12,9); $pred[d,b]$ is updated from e to a
115	f	d	c	(14,7)	(12,9)	(11,10)	e	f	
116	f	d	e	(14,7)	(12,9)	(11,10)	d	a	
117	f	e	a	(10,11)	(12,9)	(12,9)	e	f	$P_D[e,a]$ is updated from (10,11) to (12,9); $pred[e,a]$ is updated from e to f
118	f	e	b	(11,10)	(12,9)	(12,9)	e	a	$P_D[e,b]$ is updated from (11,10) to (12,9); $pred[e,b]$ is updated from e to a
119	f	e	c	(14,7)	(12,9)	(11,10)	e	f	
120	f	e	d	(13,8)	(12,9)	(11,10)	c	f	

3.6. Example 6

When $i \succsim_v j$ for every $v \in V$, then we say “alternative i Pareto-dominates alternative j”.

Suppose an alternative j is added such that:

(3.6.1) $\exists\, i \in A^{old}\ \forall\, v \in V$: $i \succsim_v^{new} j$.

(3.6.2) $\forall\, g,h \in A^{old}\ \forall\, v \in V$: $g \succ_v^{old} h \Leftrightarrow g \succ_v^{new} h$.

Then *independence from Pareto-dominated alternatives* (IPDA) says that we must get:

(3.6.3) $\forall\, g,h \in A^{old}$: $gh \in \mathcal{O}^{old} \Leftrightarrow gh \in \mathcal{O}^{new}$.

(3.6.4) $\forall\, g \in A^{old}$: $g \in \mathcal{S}^{old} \Leftrightarrow g \in \mathcal{S}^{new}$.

The following example demonstrates that the Schulze method, as defined in section 2.2, does not satisfy IPDA. This example has been proposed by Eppley (2003).

3.6.1. Situation #1

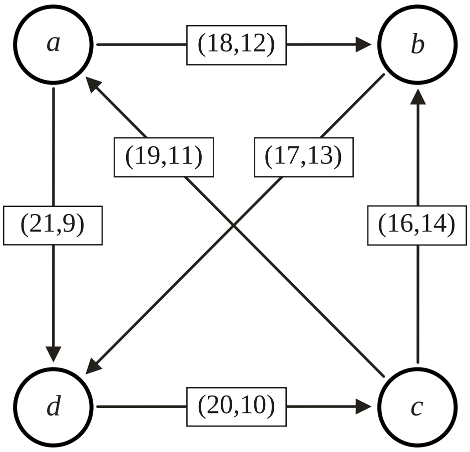
Example 6 (old):

3 voters	$a \succ_v b \succ_v d \succ_v c$
5 voters	$a \succ_v d \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
2 voters	$b \succ_v d \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d$
6 voters	$c \succ_v b \succ_v a \succ_v d$
2 voters	$d \succ_v b \succ_v c \succ_v a$
5 voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N^{old} looks as follows:

	$N^{old}[*,a]$	$N^{old}[*,b]$	$N^{old}[*,c]$	$N^{old}[*,d]$
$N^{old}[a,*]$	---	18	11	21
$N^{old}[b,*]$	12	---	14	17
$N^{old}[c,*]$	19	16	---	10
$N^{old}[d,*]$	9	13	20	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(18,12)</u>, *b*	*a*, (21,9), *d*, <u>(20,10)</u>, *c*	*a*, <u>(21,9)</u>, *d*
from *b* ...	*b*, <u>(17,13)</u>, *d*, (20,10), *c*, (19,11), *a*	---	*b*, <u>(17,13)</u>, *d*, (20,10), *c*	*b*, <u>(17,13)</u>, *d*
from *c* ...	*c*, <u>(19,11)</u>, *a*	*c*, (19,11), *a*, <u>(18,12)</u>, *b*	---	*c*, <u>(19,11)</u>, *a*, (21,9), *d*
from *d* ...	*d*, (20,10), *c*, <u>(19,11)</u>, *a*	*d*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*d*, <u>(20,10)</u>, *c*	---

We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, cb, db, dc\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(14,16)	(12,18)	(11,19)	b	a	
2	a	b	d	(17,13)	(12,18)	(21,9)	b	a	
3	a	c	b	(16,14)	(19,11)	(18,12)	c	a	$P_D[c,b]$ is updated from (16,14) to (18,12); $pred[c,b]$ is updated from c to a
4	a	c	d	(10,20)	(19,11)	(21,9)	c	a	$P_D[c,d]$ is updated from (10,20) to (19,11); $pred[c,d]$ is updated from c to a
5	a	d	b	(13,17)	(9,21)	(18,12)	d	a	
6	a	d	c	(20,10)	(9,21)	(11,19)	d	a	
7	b	a	c	(11,19)	(18,12)	(14,16)	a	b	$P_D[a,c]$ is updated from (11,19) to (14,16); $pred[a,c]$ is updated from a to b
8	b	a	d	(21,9)	(18,12)	(17,13)	a	b	
9	b	c	a	(19,11)	(18,12)	(12,18)	c	b	
10	b	c	d	(19,11)	(18,12)	(17,13)	a	b	
11	b	d	a	(9,21)	(13,17)	(12,18)	d	b	$P_D[d,a]$ is updated from (9,21) to (12,18); $pred[d,a]$ is updated from d to b
12	b	d	c	(20,10)	(13,17)	(14,16)	d	b	
13	c	a	b	(18,12)	(14,16)	(18,12)	a	a	
14	c	a	d	(21,9)	(14,16)	(19,11)	a	a	
15	c	b	a	(12,18)	(14,16)	(19,11)	b	c	$P_D[b,a]$ is updated from (12,18) to (14,16); $pred[b,a]$ is updated from b to c
16	c	b	d	(17,13)	(14,16)	(19,11)	b	a	
17	c	d	a	(12,18)	(20,10)	(19,11)	b	c	$P_D[d,a]$ is updated from (12,18) to (19,11); $pred[d,a]$ is updated from b to c
18	c	d	b	(13,17)	(20,10)	(18,12)	d	a	$P_D[d,b]$ is updated from (13,17) to (18,12); $pred[d,b]$ is updated from d to a
19	d	a	b	(18,12)	(21,9)	(18,12)	a	a	
20	d	a	c	(14,16)	(21,9)	(20,10)	b	d	$P_D[a,c]$ is updated from (14,16) to (20,10); $pred[a,c]$ is updated from b to d
21	d	b	a	(14,16)	(17,13)	(19,11)	c	c	$P_D[b,a]$ is updated from (14,16) to (17,13)
22	d	b	c	(14,16)	(17,13)	(20,10)	b	d	$P_D[b,c]$ is updated from (14,16) to (17,13); $pred[b,c]$ is updated from b to d
23	d	c	a	(19,11)	(19,11)	(19,11)	c	c	
24	d	c	b	(18,12)	(19,11)	(18,12)	a	a	

3.6.2. Situation #2

Suppose alternative *e* is added as follows:

Example 6 (new):

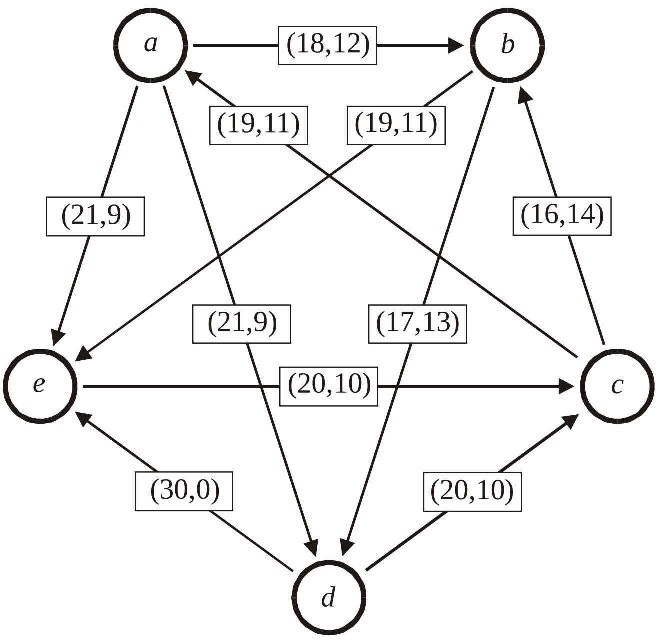
3 voters	$a \succ_v b \succ_v d \succ_v e \succ_v c$
5 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v e \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v e \succ_v c$
2 voters	$b \succ_v d \succ_v e \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d \succ_v e$
6 voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
2 voters	$d \succ_v b \succ_v e \succ_v c \succ_v a$
5 voters	$d \succ_v e \succ_v c \succ_v a \succ_v b$

The newly added alternative *e* is Pareto-dominated by alternative *d*, because $d \succ_v e$ for every voter $v \in V$. Therefore, (3.6.1) – (3.6.4) say that the result should not change.

The pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$
$N^{new}[a,*]$	---	18	11	21	21
$N^{new}[b,*]$	12	---	14	17	19
$N^{new}[c,*]$	19	16	---	10	10
$N^{new}[d,*]$	9	13	20	---	30
$N^{new}[e,*]$	9	11	20	0	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, $\underline{(18,12)}$, *b*	*a*, (21,9), *d*, $\underline{(20,10)}$, *c*	*a*, $\underline{(21,9)}$, *d*	*a*, $\underline{(21,9)}$, *e*
from *b* ...	*b*, $\underline{(19,11)}$, *e*, (20,10), *c*, $\underline{(19,11)}$, *a*	---	*b*, $\underline{(19,11)}$, *e*, (20,10), *c*	*b*, $\underline{(19,11)}$, *e*, (20,10), *c*, $\underline{(19,11)}$, *a*, (21,9), *d*	*b*, $\underline{(19,11)}$, *e*
from *c* ...	*c*, $\underline{(19,11)}$, *a*	*c*, (19,11), *a*, $\underline{(18,12)}$, *b*	---	*c*, $\underline{(19,11)}$, *a*, (21,9), *d*	*c*, $\underline{(19,11)}$, *a*, (21,9), *e*
from *d* ...	*d*, (20,10), *c*, $\underline{(19,11)}$, *a*	*d*, (20,10), *c*, (19,11), *a*, $\underline{(18,12)}$, *b*	*d*, $\underline{(20,10)}$, *c*	---	*d*, $\underline{(30,0)}$, *e*
from *e* ...	*e*, (20,10), *c*, $\underline{(19,11)}$, *a*	*e*, (20,10), *c*, (19,11), *a*, $\underline{(18,12)}$, *b*	*e*, $\underline{(20,10)}$, *c*	*e*, (20,10), *c*, $\underline{(19,11)}$, *a*, (21,9), *d*	---

We get $\mathcal{O}^{\text{new}} = \{ac, ad, ae, ba, bc, bd, be, dc, de, ec\}$ and $\mathcal{S}^{\text{new}} = \{b\}$.

Example 6 shows that the Schulze method, as defined in section 2.2, violates IPDA, as defined in (3.6.1) – (3.6.4). For example, we have (1) $ab \in \mathcal{O}^{\text{old}}$ and $ba \in \mathcal{O}^{\text{new}}$, (2) $cb \in \mathcal{O}^{\text{old}}$ and $bc \in \mathcal{O}^{\text{new}}$, (3) $db \in \mathcal{O}^{\text{old}}$ and $bd \in \mathcal{O}^{\text{new}}$, (4) $a \in \mathcal{S}^{\text{old}}$ and $a \notin \mathcal{S}^{\text{new}}$, and (5) $b \notin \mathcal{S}^{\text{old}}$ and $b \in \mathcal{S}^{\text{new}}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C\text{–}1) \cdot (C\text{–}2) = 60$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(14,16)	(12,18)	(11,19)	b	a	
2	a	b	d	(17,13)	(12,18)	(21,9)	b	a	
3	a	b	e	(19,11)	(12,18)	(21,9)	b	a	
4	a	c	b	(16,14)	(19,11)	(18,12)	c	a	$P_D[c,b]$ is updated from (16,14) to (18,12); $pred[c,b]$ is updated from c to a
5	a	c	d	(10,20)	(19,11)	(21,9)	c	a	$P_D[c,d]$ is updated from (10,20) to (19,11); $pred[c,d]$ is updated from c to a
6	a	c	e	(10,20)	(19,11)	(21,9)	c	a	$P_D[c,e]$ is updated from (10,20) to (19,11); $pred[c,e]$ is updated from c to a
7	a	d	b	(13,17)	(9,21)	(18,12)	d	a	
8	a	d	c	(20,10)	(9,21)	(11,19)	d	a	
9	a	d	e	(30,0)	(9,21)	(21,9)	d	a	
10	a	e	b	(11,19)	(9,21)	(18,12)	e	a	
11	a	e	c	(20,10)	(9,21)	(11,19)	e	a	
12	a	e	d	(0,30)	(9,21)	(21,9)	e	a	$P_D[e,d]$ is updated from (0,30) to (9,21); $pred[e,d]$ is updated from e to a
13	b	a	c	(11,19)	(18,12)	(14,16)	a	b	$P_D[a,c]$ is updated from (11,19) to (14,16); $pred[a,c]$ is updated from a to b
14	b	a	d	(21,9)	(18,12)	(17,13)	a	b	
15	b	a	e	(21,9)	(18,12)	(19,11)	a	b	
16	b	c	a	(19,11)	(18,12)	(12,18)	c	b	
17	b	c	d	(19,11)	(18,12)	(17,13)	a	b	
18	b	c	e	(19,11)	(18,12)	(19,11)	a	b	
19	b	d	a	(9,21)	(13,17)	(12,18)	d	b	$P_D[d,a]$ is updated from (9,21) to (12,18); $pred[d,a]$ is updated from d to b
20	b	d	c	(20,10)	(13,17)	(14,16)	d	b	
21	b	d	e	(30,0)	(13,17)	(19,11)	d	b	
22	b	e	a	(9,21)	(11,19)	(12,18)	e	b	$P_D[e,a]$ is updated from (9,21) to (11,19); $pred[e,a]$ is updated from e to b
23	b	e	c	(20,10)	(11,19)	(14,16)	e	b	
24	b	e	d	(9,21)	(11,19)	(17,13)	a	b	$P_D[e,d]$ is updated from (9,21) to (11,19); $pred[e,d]$ is updated from a to b
25	c	a	b	(18,12)	(14,16)	(18,12)	a	a	
26	c	a	d	(21,9)	(14,16)	(19,11)	a	a	
27	c	a	e	(21,9)	(14,16)	(19,11)	a	a	
28	c	b	a	(12,18)	(14,16)	(19,11)	b	c	$P_D[b,a]$ is updated from (12,18) to (14,16); $pred[b,a]$ is updated from b to c
29	c	b	d	(17,13)	(14,16)	(19,11)	b	a	
30	c	b	e	(19,11)	(14,16)	(19,11)	b	a	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(12,18)	(20,10)	(19,11)	b	c	$P_D[d,a]$ is updated from (12,18) to (19,11); $pred[d,a]$ is updated from b to c
32	c	d	b	(13,17)	(20,10)	(18,12)	d	a	$P_D[d,b]$ is updated from (13,17) to (18,12); $pred[d,b]$ is updated from d to a
33	c	d	e	(30,0)	(20,10)	(19,11)	d	a	
34	c	e	a	(11,19)	(20,10)	(19,11)	b	c	$P_D[e,a]$ is updated from (11,19) to (19,11); $pred[e,a]$ is updated from b to c
35	c	e	b	(11,19)	(20,10)	(18,12)	e	a	$P_D[e,b]$ is updated from (11,19) to (18,12); $pred[e,b]$ is updated from e to a
36	c	e	d	(11,19)	(20,10)	(19,11)	b	a	$P_D[e,d]$ is updated from (11,19) to (19,11); $pred[e,d]$ is updated from b to a
37	d	a	b	(18,12)	(21,9)	(18,12)	a	a	
38	d	a	c	(14,16)	(21,9)	(20,10)	b	d	$P_D[a,c]$ is updated from (14,16) to (20,10); $pred[a,c]$ is updated from b to d
39	d	a	e	(21,9)	(21,9)	(30,0)	a	d	
40	d	b	a	(14,16)	(17,13)	(19,11)	c	c	$P_D[b,a]$ is updated from (14,16) to (17,13)
41	d	b	c	(14,16)	(17,13)	(20,10)	b	d	$P_D[b,c]$ is updated from (14,16) to (17,13); $pred[b,c]$ is updated from b to d
42	d	b	e	(19,11)	(17,13)	(30,0)	b	d	
43	d	c	a	(19,11)	(19,11)	(19,11)	c	c	
44	d	c	b	(18,12)	(19,11)	(18,12)	a	a	
45	d	c	e	(19,11)	(19,11)	(30,0)	a	d	
46	d	e	a	(19,11)	(19,11)	(19,11)	c	c	
47	d	e	b	(18,12)	(19,11)	(18,12)	a	a	
48	d	e	c	(20,10)	(19,11)	(20,10)	e	d	
49	e	a	b	(18,12)	(21,9)	(18,12)	a	a	
50	e	a	c	(20,10)	(21,9)	(20,10)	d	e	
51	e	a	d	(21,9)	(21,9)	(19,11)	a	a	
52	e	b	a	(17,13)	(19,11)	(19,11)	c	c	$P_D[b,a]$ is updated from (17,13) to (19,11)
53	e	b	c	(17,13)	(19,11)	(20,10)	d	e	$P_D[b,c]$ is updated from (17,13) to (19,11); $pred[b,c]$ is updated from d to e
54	e	b	d	(17,13)	(19,11)	(19,11)	b	a	$P_D[b,d]$ is updated from (17,13) to (19,11); $pred[b,d]$ is updated from b to a
55	e	c	a	(19,11)	(19,11)	(19,11)	c	c	
56	e	c	b	(18,12)	(19,11)	(18,12)	a	a	
57	e	c	d	(19,11)	(19,11)	(19,11)	a	a	
58	e	d	a	(19,11)	(30,0)	(19,11)	c	c	
59	e	d	b	(18,12)	(30,0)	(18,12)	a	a	
60	e	d	c	(20,10)	(30,0)	(20,10)	d	e	

3.7. Example 7

3.7.1. Situation #1

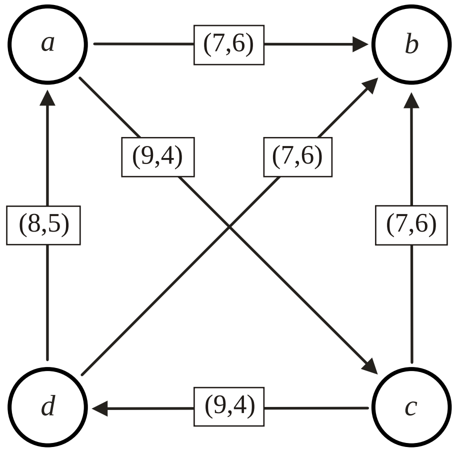
Example 7 (old):

5 voters	$a \succ_v c \succ_v d \succ_v b$
2 voters	$b \succ_v c \succ_v d \succ_v a$
4 voters	$b \succ_v d \succ_v a \succ_v c$
2 voters	$c \succ_v d \succ_v a \succ_v b$

The pairwise matrix N^{old} looks as follows:

	$N^{old}[*,a]$	$N^{old}[*,b]$	$N^{old}[*,c]$	$N^{old}[*,d]$
$N^{old}[a,*]$	---	7	9	5
$N^{old}[b,*]$	6	---	6	6
$N^{old}[c,*]$	4	7	---	9
$N^{old}[d,*]$	8	7	4	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(7,6)</u>, *b*	*a*, <u>(9,4)</u>, *c*	*a*, <u>(9,4)</u>, *c*, <u>(9,4)</u>, *d*
from *b* ...	*b*, <u>(6,7)</u>, *a*	---	*b*, <u>(6,7)</u>, *c*	*b*, <u>(6,7)</u>, *d*
from *c* ...	*c*, (9,4), *d*, <u>(8,5)</u>, *a*	*c*, <u>(7,6)</u>, *b*	---	*c*, <u>(9,4)</u>, *d*
from *d* ...	*d*, <u>(8,5)</u>, *a*	*d*, <u>(7,6)</u>, *b*	*d*, <u>(8,5)</u>, *a*, (9,4), *c*	---

We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, cb, cd, db\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(6,7)	(6,7)	(9,4)	b	a	
2	a	b	d	(6,7)	(6,7)	(5,8)	b	a	
3	a	c	b	(7,6)	(4,9)	(7,6)	c	a	
4	a	c	d	(9,4)	(4,9)	(5,8)	c	a	
5	a	d	b	(7,6)	(8,5)	(7,6)	d	a	
6	a	d	c	(4,9)	(8,5)	(9,4)	d	a	$P_D[d,c]$ is updated from (4,9) to (8,5); $pred[d,c]$ is updated from d to a
7	b	a	c	(9,4)	(7,6)	(6,7)	a	b	
8	b	a	d	(5,8)	(7,6)	(6,7)	a	b	$P_D[a,d]$ is updated from (5,8) to (6,7); $pred[a,d]$ is updated from a to b
9	b	c	a	(4,9)	(7,6)	(6,7)	c	b	$P_D[c,a]$ is updated from (4,9) to (6,7); $pred[c,a]$ is updated from c to b
10	b	c	d	(9,4)	(7,6)	(6,7)	c	b	
11	b	d	a	(8,5)	(7,6)	(6,7)	d	b	
12	b	d	c	(8,5)	(7,6)	(6,7)	a	b	
13	c	a	b	(7,6)	(9,4)	(7,6)	a	c	
14	c	a	d	(6,7)	(9,4)	(9,4)	b	c	$P_D[a,d]$ is updated from (6,7) to (9,4); $pred[a,d]$ is updated from b to c
15	c	b	a	(6,7)	(6,7)	(6,7)	b	b	
16	c	b	d	(6,7)	(6,7)	(9,4)	b	c	
17	c	d	a	(8,5)	(8,5)	(6,7)	d	b	
18	c	d	b	(7,6)	(8,5)	(7,6)	d	c	
19	d	a	b	(7,6)	(9,4)	(7,6)	a	d	
20	d	a	c	(9,4)	(9,4)	(8,5)	a	a	
21	d	b	a	(6,7)	(6,7)	(8,5)	b	d	
22	d	b	c	(6,7)	(6,7)	(8,5)	b	a	
23	d	c	a	(6,7)	(9,4)	(8,5)	b	d	$P_D[c,a]$ is updated from (6,7) to (8,5); $pred[c,a]$ is updated from b to d
24	d	c	b	(7,6)	(9,4)	(7,6)	c	d	

3.7.2. Situation #2

Suppose alternative e is added as follows:

Example 7 (new):

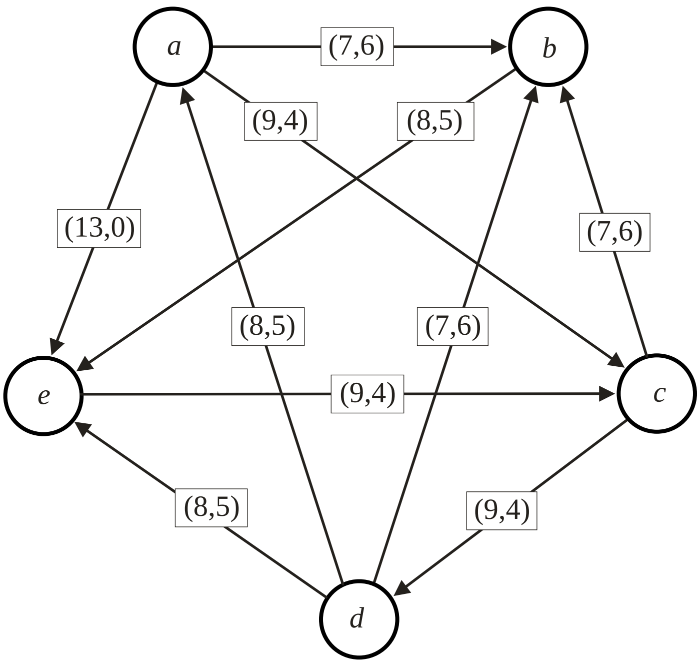
5 voters	$a \succ_v e \succ_v c \succ_v d \succ_v b$
2 voters	$b \succ_v c \succ_v d \succ_v a \succ_v e$
4 voters	$b \succ_v d \succ_v a \succ_v e \succ_v c$
2 voters	$c \succ_v d \succ_v a \succ_v b \succ_v e$

The newly added alternative e is Pareto-dominated by alternative a, because $a \succ_v e$ for every voter $v \in V$. Therefore, (3.6.1) – (3.6.4) say that the result should not change.

The pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$
$N^{new}[a,*]$	---	7	9	5	13
$N^{new}[b,*]$	6	---	6	6	8
$N^{new}[c,*]$	4	7	---	9	4
$N^{new}[d,*]$	8	7	4	---	8
$N^{new}[e,*]$	0	5	9	5	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, $\underline{(7,6)}$, *b*	*a*, $\underline{(9,4)}$, *c*	*a*, $\underline{(9,4)}$, *c*, $\underline{(9,4)}$, *d*	*a*, $\underline{(13,0)}$, *e*
from *b* ...	*b*, $\underline{(8,5)}$, *e*, (9,4), *c*, (9,4), *d*, $\underline{(8,5)}$, *a*	---	*b*, $\underline{(8,5)}$, *e*, (9,4), *c*	*b*, $\underline{(8,5)}$, *e*, (9,4), *c*, (9,4), *d*	*b*, $\underline{(8,5)}$, *e*
from *c* ...	*c*, (9,4), *d*, $\underline{(8,5)}$, *a*	*c*, $\underline{(7,6)}$, *b*	---	*c*, $\underline{(9,4)}$, *d*	*c*, (9,4), *d*, $\underline{(8,5)}$, *e*
from *d* ...	*d*, $\underline{(8,5)}$, *a*	*d*, $\underline{(7,6)}$, *b*	*d*, $\underline{(8,5)}$, *a*, (9,4), *c*	---	*d*, $\underline{(8,5)}$, *e*
from *e* ...	*e*, (9,4), *c*, (9,4), *d*, $\underline{(8,5)}$, *a*	*e*, (9,4), *c*, $\underline{(7,6)}$, *b*	*e*, $\underline{(9,4)}$, *c*	*e*, $\underline{(9,4)}$, *c*, $\underline{(9,4)}$, *d*	---

We get $\mathcal{O}^{new} = \{ac, ad, ae, ba, bc, bd, be, cd, ec, ed\}$ and $\mathcal{S}^{new} = \{b\}$.

Example 7 shows that the Schulze method, as defined in section 2.2, violates IPDA, as defined in (3.6.1) – (3.6.4). For example, we have (1) $ab \in \mathcal{O}^{old}$ and $ba \in \mathcal{O}^{new}$, (2) $cb \in \mathcal{O}^{old}$ and $bc \in \mathcal{O}^{new}$, (3) $db \in \mathcal{O}^{old}$ and $bd \in \mathcal{O}^{new}$, (4) $a \in \mathcal{S}^{old}$ and $a \notin \mathcal{S}^{new}$, and (5) $b \notin \mathcal{S}^{old}$ and $b \in \mathcal{S}^{new}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 60$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(6,7)	(6,7)	(9,4)	b	a	
2	a	b	d	(6,7)	(6,7)	(5,8)	b	a	
3	a	b	e	(8,5)	(6,7)	(13,0)	b	a	
4	a	c	b	(7,6)	(4,9)	(7,6)	c	a	
5	a	c	d	(9,4)	(4,9)	(5,8)	c	a	
6	a	c	e	(4,9)	(4,9)	(13,0)	c	a	
7	a	d	b	(7,6)	(8,5)	(7,6)	d	a	
8	a	d	c	(4,9)	(8,5)	(9,4)	d	a	$P_D[d,c]$ is updated from (4,9) to (8,5); *pred*[*d*,*c*] is updated from *d* to *a*
9	a	d	e	(8,5)	(8,5)	(13,0)	d	a	
10	a	e	b	(5,8)	(0,13)	(7,6)	e	a	
11	a	e	c	(9,4)	(0,13)	(9,4)	e	a	
12	a	e	d	(5,8)	(0,13)	(5,8)	e	a	
13	b	a	c	(9,4)	(7,6)	(6,7)	a	b	
14	b	a	d	(5,8)	(7,6)	(6,7)	a	b	$P_D[a,d]$ is updated from (5,8) to (6,7); *pred*[*a*,*d*] is updated from *a* to *b*
15	b	a	e	(13,0)	(7,6)	(8,5)	a	b	
16	b	c	a	(4,9)	(7,6)	(6,7)	c	b	$P_D[c,a]$ is updated from (4,9) to (6,7); *pred*[*c*,*a*] is updated from *c* to *b*
17	b	c	d	(9,4)	(7,6)	(6,7)	c	b	
18	b	c	e	(4,9)	(7,6)	(8,5)	c	b	$P_D[c,e]$ is updated from (4,9) to (7,6); *pred*[*c*,*e*] is updated from *c* to *b*
19	b	d	a	(8,5)	(7,6)	(6,7)	d	b	
20	b	d	c	(8,5)	(7,6)	(6,7)	a	b	
21	b	d	e	(8,5)	(7,6)	(8,5)	d	b	
22	b	e	a	(0,13)	(5,8)	(6,7)	e	b	$P_D[e,a]$ is updated from (0,13) to (5,8); *pred*[*e*,*a*] is updated from *e* to *b*
23	b	e	c	(9,4)	(5,8)	(6,7)	e	b	
24	b	e	d	(5,8)	(5,8)	(6,7)	e	b	
25	c	a	b	(7,6)	(9,4)	(7,6)	a	c	
26	c	a	d	(6,7)	(9,4)	(9,4)	b	c	$P_D[a,d]$ is updated from (6,7) to (9,4); *pred*[*a*,*d*] is updated from *b* to *c*
27	c	a	e	(13,0)	(9,4)	(7,6)	a	b	
28	c	b	a	(6,7)	(6,7)	(6,7)	b	b	
29	c	b	d	(6,7)	(6,7)	(9,4)	b	c	
30	c	b	e	(8,5)	(6,7)	(7,6)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(8,5)	(8,5)	(6,7)	d	b	
32	c	d	b	(7,6)	(8,5)	(7,6)	d	c	
33	c	d	e	(8,5)	(8,5)	(7,6)	d	b	
34	c	e	a	(5,8)	(9,4)	(6,7)	b	b	$P_D[e,a]$ is updated from (5,8) to (6,7)
35	c	e	b	(5,8)	(9,4)	(7,6)	e	c	$P_D[e,b]$ is updated from (5,8) to (7,6); $pred[e,b]$ is updated from e to c
36	c	e	d	(5,8)	(9,4)	(9,4)	e	c	$P_D[e,d]$ is updated from (5,8) to (9,4); $pred[e,d]$ is updated from e to c
37	d	a	b	(7,6)	(9,4)	(7,6)	a	d	
38	d	a	c	(9,4)	(9,4)	(8,5)	a	a	
39	d	a	e	(13,0)	(9,4)	(8,5)	a	d	
40	d	b	a	(6,7)	(6,7)	(8,5)	b	d	
41	d	b	c	(6,7)	(6,7)	(8,5)	b	a	
42	d	b	e	(8,5)	(6,7)	(8,5)	b	d	
43	d	c	a	(6,7)	(9,4)	(8,5)	b	d	$P_D[c,a]$ is updated from (6,7) to (8,5); $pred[c,a]$ is updated from b to d
44	d	c	b	(7,6)	(9,4)	(7,6)	c	d	
45	d	c	e	(7,6)	(9,4)	(8,5)	b	d	$P_D[c,e]$ is updated from (7,6) to (8,5); $pred[c,e]$ is updated from b to d
46	d	e	a	(6,7)	(9,4)	(8,5)	b	d	$P_D[e,a]$ is updated from (6,7) to (8,5); $pred[e,a]$ is updated from b to d
47	d	e	b	(7,6)	(9,4)	(7,6)	c	d	
48	d	e	c	(9,4)	(9,4)	(8,5)	e	a	
49	e	a	b	(7,6)	(13,0)	(7,6)	a	c	
50	e	a	c	(9,4)	(13,0)	(9,4)	a	e	
51	e	a	d	(9,4)	(13,0)	(9,4)	c	c	
52	e	b	a	(6,7)	(8,5)	(8,5)	b	d	$P_D[b,a]$ is updated from (6,7) to (8,5); $pred[b,a]$ is updated from b to d
53	e	b	c	(6,7)	(8,5)	(9,4)	b	e	$P_D[b,c]$ is updated from (6,7) to (8,5); $pred[b,c]$ is updated from b to e
54	e	b	d	(6,7)	(8,5)	(9,4)	b	c	$P_D[b,d]$ is updated from (6,7) to (8,5); $pred[b,d]$ is updated from b to c
55	e	c	a	(8,5)	(8,5)	(8,5)	d	d	
56	e	c	b	(7,6)	(8,5)	(7,6)	c	c	
57	e	c	d	(9,4)	(8,5)	(9,4)	c	c	
58	e	d	a	(8,5)	(8,5)	(8,5)	d	d	
59	e	d	b	(7,6)	(8,5)	(7,6)	d	c	
60	e	d	c	(8,5)	(8,5)	(9,4)	a	e	

3.8. Example 8

When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy presumption (2.1.1), are equivalent. However, when some voters cast non-linear orders, then there are many possible definitions for the strength of a link. The following example illustrates how the different definitions for the strength of a link can lead to different winners.

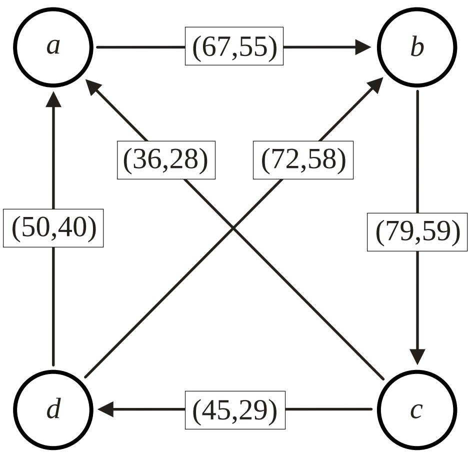
Example 8:

6	voters	$a \succ_v b \succ_v c \succ_v d$
8	voters	$a \approx_v b \succ_v c \approx_v d$
8	voters	$a \approx_v c \succ_v b \approx_v d$
18	voters	$a \approx_v c \succ_v d \succ_v b$
8	voters	$a \approx_v c \approx_v d \succ_v b$
40	voters	$b \succ_v a \approx_v c \approx_v d$
4	voters	$c \succ_v b \succ_v d \succ_v a$
9	voters	$c \succ_v d \succ_v a \succ_v b$
8	voters	$c \approx_v d \succ_v a \approx_v b$
14	voters	$d \succ_v a \succ_v b \succ_v c$
11	voters	$d \succ_v b \succ_v c \succ_v a$
4	voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	67	28	40
$N[b,*]$	55	---	79	58
$N[c,*]$	36	59	---	45
$N[d,*]$	50	72	29	---

The corresponding digraph looks as follows:



a) margin

We get: $(N[b,c],N[c,b]) \succ_{margin} (N[c,d],N[d,c]) \succ_{margin} (N[d,b],N[b,d]) \succ_{margin} (N[a,b],N[b,a]) \succ_{margin} (N[d,a],N[a,d]) \succ_{margin} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a margin of $N[b,c] - N[c,b] = 20$
cd with a margin of $N[c,d] - N[d,c] = 16$
db with a margin of $N[d,b] - N[b,d] = 14$
ab with a margin of $N[a,b] - N[b,a] = 12$
da with a margin of $N[d,a] - N[a,d] = 10$
ca with a margin of $N[c,a] - N[a,c] = 8$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, (45,29), *d*, <u>(50,40)</u>, *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, (79,59), *c*, <u>(45,29)</u>, *d*
from *c* ...	*c*, (45,29), *d*, <u>(50,40)</u>, *a*	*c*, (45,29), *d*, <u>(72,58)</u>, *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, <u>(72,58)</u>, *b*	*d*, <u>(72,58)</u>, *b*, (79,59), *c*	---

We get $\mathcal{O}_{margin} = \{ab, ac, ad, bc, bd, cd\}$ and $\mathcal{S}_{margin} = \{a\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C-1) \cdot (C-2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_{margin}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{margin}[j,k]$	$P_{margin}[j,i]$	$P_{margin}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	$P_{margin}[b,d]$ is updated from (58,72) to (55,67); $pred[b,d]$ is updated from b to a
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{margin}[c,b]$ is updated from (59,79) to (36,28); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	$P_{margin}[d,c]$ is updated from (29,45) to (28,36); $pred[d,c]$ is updated from d to a
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{margin}[a,c]$ is updated from (28,36) to (67,55); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(55,67)	a	a	
9	b	c	a	(36,28)	(36,28)	(55,67)	c	b	
10	b	c	d	(45,29)	(36,28)	(55,67)	c	a	
11	b	d	a	(50,40)	(72,58)	(55,67)	d	b	
12	b	d	c	(28,36)	(72,58)	(79,59)	a	b	$P_{margin}[d,c]$ is updated from (28,36) to (72,58); $pred[d,c]$ is updated from a to b
13	c	a	b	(67,55)	(67,55)	(36,28)	a	a	
14	c	a	d	(40,50)	(67,55)	(45,29)	a	c	$P_{margin}[a,d]$ is updated from (40,50) to (67,55); $pred[a,d]$ is updated from a to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{margin}[b,a]$ is updated from (55,67) to (36,28); $pred[b,a]$ is updated from b to c
16	c	b	d	(55,67)	(79,59)	(45,29)	a	c	$P_{margin}[b,d]$ is updated from (55,67) to (45,29); $pred[b,d]$ is updated from a to c
17	c	d	a	(50,40)	(72,58)	(36,28)	d	c	
18	c	d	b	(72,58)	(72,58)	(36,28)	d	a	
19	d	a	b	(67,55)	(67,55)	(72,58)	a	d	
20	d	a	c	(67,55)	(67,55)	(72,58)	b	b	
21	d	b	a	(36,28)	(45,29)	(50,40)	c	d	$P_{margin}[b,a]$ is updated from (36,28) to (50,40); $pred[b,a]$ is updated from c to d
22	d	b	c	(79,59)	(45,29)	(72,58)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	$P_{margin}[c,a]$ is updated from (36,28) to (50,40); $pred[c,a]$ is updated from c to d
24	d	c	b	(36,28)	(45,29)	(72,58)	a	d	$P_{margin}[c,b]$ is updated from (36,28) to (72,58); $pred[c,b]$ is updated from a to d

b) ratio

We get: $(N[c,d],N[d,c]) \succ_{ratio} (N[b,c],N[c,b]) \succ_{ratio} (N[c,a],N[a,c]) \succ_{ratio} (N[d,a],N[a,d]) \succ_{ratio} (N[d,b],N[b,d]) \succ_{ratio} (N[a,b],N[b,a])$.

The pairwise victories are:

cd with a ratio of $N[c,d] / N[d,c] = 1.552$
bc with a ratio of $N[b,c] / N[c,b] = 1.339$
ca with a ratio of $N[c,a] / N[a,c] = 1.286$
da with a ratio of $N[d,a] / N[a,d] = 1.250$
db with a ratio of $N[d,b] / N[b,d] = 1.241$
ab with a ratio of $N[a,b] / N[b,a] = 1.218$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, <u>(36,28)</u>, *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *c* ...	*c*, <u>(36,28)</u>, *a*	*c*, (45,29), *d*, <u>(72,58)</u>, *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, <u>(72,58)</u>, *b*	*d*, <u>(72,58)</u>, *b*, (79,59), *c*	---

We get $\mathcal{O}_{ratio} = \{ba, bc, bd, ca, cd, da\}$ and $\mathcal{S}_{ratio} = \{b\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_{ratio}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{ratio}[j,k]$	$P_{ratio}[j,i]$	$P_{ratio}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{ratio}[c,b]$ is updated from (59,79) to (67,55); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	$P_{ratio}[d,c]$ is updated from (29,45) to (28,36); $pred[d,c]$ is updated from d to a
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{ratio}[a,c]$ is updated from (28,36) to (67,55); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(58,72)	a	b	$P_{ratio}[a,d]$ is updated from (40,50) to (58,72); $pred[a,d]$ is updated from a to b
9	b	c	a	(36,28)	(67,55)	(55,67)	c	b	
10	b	c	d	(45,29)	(67,55)	(58,72)	c	b	
11	b	d	a	(50,40)	(72,58)	(55,67)	d	b	
12	b	d	c	(28,36)	(72,58)	(79,59)	a	b	$P_{ratio}[d,c]$ is updated from (28,36) to (72,58); $pred[d,c]$ is updated from a to b
13	c	a	b	(67,55)	(67,55)	(67,55)	a	a	
14	c	a	d	(58,72)	(67,55)	(45,29)	b	c	$P_{ratio}[a,d]$ is updated from (58,72) to (67,55); $pred[a,d]$ is updated from b to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{ratio}[b,a]$ is updated from (55,67) to (36,28); $pred[b,a]$ is updated from b to c
16	c	b	d	(58,72)	(79,59)	(45,29)	b	c	$P_{ratio}[b,d]$ is updated from (58,72) to (79,59); $pred[b,d]$ is updated from b to c
17	c	d	a	(50,40)	(72,58)	(36,28)	d	c	
18	c	d	b	(72,58)	(72,58)	(67,55)	d	a	
19	d	a	b	(67,55)	(67,55)	(72,58)	a	d	
20	d	a	c	(67,55)	(67,55)	(72,58)	b	b	
21	d	b	a	(36,28)	(79,59)	(50,40)	c	d	
22	d	b	c	(79,59)	(79,59)	(72,58)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	
24	d	c	b	(67,55)	(45,29)	(72,58)	a	d	$P_{ratio}[c,b]$ is updated from (67,55) to (72,58); $pred[c,b]$ is updated from a to d

c) winning votes

We get: $(N[b,c],N[c,b]) \succ_{win} (N[d,b],N[b,d]) \succ_{win} (N[a,b],N[b,a]) \succ_{win} (N[d,a],N[a,d]) \succ_{win} (N[c,d],N[d,c]) \succ_{win} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a support of $N[b,c] = 79$
db with a support of $N[d,b] = 72$
ab with a support of $N[a,b] = 67$
da with a support of $N[d,a] = 50$
cd with a support of $N[c,d] = 45$
ca with a support of $N[c,a] = 36$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, <u>(67,55)</u>, *b*, (79,59), *c*	*a*, (67,55), *b*, (79,59), *c*, <u>(45,29)</u>, *d*
from *b* ...	*b*, (79,59), *c*, <u>(45,29)</u>, *d*, (50,40), *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, (79,59), *c*, <u>(45,29)</u>, *d*
from *c* ...	*c*, <u>(45,29)</u>, *d*, (50,40), *a*	*c*, <u>(45,29)</u>, *d*, (72,58), *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, <u>(72,58)</u>, *b*	*d*, <u>(72,58)</u>, *b*, (79,59), *c*	---

We get $\mathcal{O}_{win} = \{ab, ac, bc, da, db, dc\}$ and $\mathcal{S}_{win} = \{d\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_{win}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{win}[j,k]$	$P_{win}[j,i]$	$P_{win}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	$P_{win}[b,d]$ is updated from (58,72) to (55,67); $pred[b,d]$ is updated from b to a
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{win}[c,b]$ is updated from (59,79) to (36,28); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	$P_{win}[d,c]$ is updated from (29,45) to (28,36); $pred[d,c]$ is updated from d to a
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{win}[a,c]$ is updated from (28,36) to (67,55); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(55,67)	a	a	
9	b	c	a	(36,28)	(36,28)	(55,67)	c	b	
10	b	c	d	(45,29)	(36,28)	(55,67)	c	a	
11	b	d	a	(50,40)	(72,58)	(55,67)	d	b	
12	b	d	c	(28,36)	(72,58)	(79,59)	a	b	$P_{win}[d,c]$ is updated from (28,36) to (72,58); $pred[d,c]$ is updated from a to b
13	c	a	b	(67,55)	(67,55)	(36,28)	a	a	
14	c	a	d	(40,50)	(67,55)	(45,29)	a	c	$P_{win}[a,d]$ is updated from (40,50) to (45,29); $pred[a,d]$ is updated from a to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{win}[b,a]$ is updated from (55,67) to (36,28); $pred[b,a]$ is updated from b to c
16	c	b	d	(55,67)	(79,59)	(45,29)	a	c	$P_{win}[b,d]$ is updated from (55,67) to (45,29); $pred[b,d]$ is updated from a to c
17	c	d	a	(50,40)	(72,58)	(36,28)	d	c	
18	c	d	b	(72,58)	(72,58)	(36,28)	d	a	
19	d	a	b	(67,55)	(45,29)	(72,58)	a	d	
20	d	a	c	(67,55)	(45,29)	(72,58)	b	b	
21	d	b	a	(36,28)	(45,29)	(50,40)	c	d	$P_{win}[b,a]$ is updated from (36,28) to (45,29); $pred[b,a]$ is updated from c to d
22	d	b	c	(79,59)	(45,29)	(72,58)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	$P_{win}[c,a]$ is updated from (36,28) to (45,29); $pred[c,a]$ is updated from c to d
24	d	c	b	(36,28)	(45,29)	(72,58)	a	d	$P_{win}[c,b]$ is updated from (36,28) to (45,29); $pred[c,b]$ is updated from a to d

d) losing votes

We get: $(N[c,a],N[a,c]) \succ_{los} (N[c,d],N[d,c]) \succ_{los} (N[d,a],N[a,d]) \succ_{los} (N[a,b],N[b,a]) \succ_{los} (N[d,b],N[b,d]) \succ_{los} (N[b,c],N[c,b])$.

The pairwise victories are:

ca with an opposition of $N[a,c] = 28$
cd with an opposition of $N[d,c] = 29$
da with an opposition of $N[a,d] = 40$
ab with an opposition of $N[b,a] = 55$
db with an opposition of $N[b,d] = 58$
bc with an opposition of $N[c,b] = 59$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, <u>(67,55)</u>, *b*	*a*, (67,55), *b*, <u>(79,59)</u>, *c*	*a*, (67,55), *b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *b* ...	*b*, <u>(79,59)</u>, *c*, (36,28), *a*	---	*b*, <u>(79,59)</u>, *c*	*b*, <u>(79,59)</u>, *c*, (45,29), *d*
from *c* ...	*c*, <u>(36,28)</u>, *a*	*c*, (36,28), *a*, <u>(67,55)</u>, *b*	---	*c*, <u>(45,29)</u>, *d*
from *d* ...	*d*, <u>(50,40)</u>, *a*	*d*, (50,40), *a* <u>(67,55)</u>, *b*	*d*, (50,40), *a* (67,55), *b*, <u>(79,59)</u>, *c*	---

We get $\mathcal{O}_{los} = \{ab, ca, cb, cd, da, db\}$ and $\mathcal{S}_{los} = \{c\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C-1) \cdot (C-2) = 24$ steps of the Floyd-Warshall algorithm.

We start with

- $P_{los}[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.

- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_{los}[j,k]$	$P_{los}[j,i]$	$P_{los}[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(79,59)	(55,67)	(28,36)	b	a	
2	a	b	d	(58,72)	(55,67)	(40,50)	b	a	
3	a	c	b	(59,79)	(36,28)	(67,55)	c	a	$P_{los}[c,b]$ is updated from (59,79) to (67,55); $pred[c,b]$ is updated from c to a
4	a	c	d	(45,29)	(36,28)	(40,50)	c	a	
5	a	d	b	(72,58)	(50,40)	(67,55)	d	a	$P_{los}[d,b]$ is updated from (72,58) to (67,55); $pred[d,b]$ is updated from d to a
6	a	d	c	(29,45)	(50,40)	(28,36)	d	a	
7	b	a	c	(28,36)	(67,55)	(79,59)	a	b	$P_{los}[a,c]$ is updated from (28,36) to (79,59); $pred[a,c]$ is updated from a to b
8	b	a	d	(40,50)	(67,55)	(58,72)	a	b	$P_{los}[a,d]$ is updated from (40,50) to (58,72); $pred[a,d]$ is updated from a to b
9	b	c	a	(36,28)	(67,55)	(55,67)	c	b	
10	b	c	d	(45,29)	(67,55)	(58,72)	c	b	
11	b	d	a	(50,40)	(67,55)	(55,67)	d	b	
12	b	d	c	(29,45)	(67,55)	(79,59)	d	b	$P_{los}[d,c]$ is updated from (29,45) to (79,59); $pred[d,c]$ is updated from d to b
13	c	a	b	(67,55)	(79,59)	(67,55)	a	a	
14	c	a	d	(58,72)	(79,59)	(45,29)	b	c	$P_{los}[a,d]$ is updated from (58,72) to (79,59); $pred[a,d]$ is updated from b to c
15	c	b	a	(55,67)	(79,59)	(36,28)	b	c	$P_{los}[b,a]$ is updated from (55,67) to (79,59); $pred[b,a]$ is updated from b to c
16	c	b	d	(58,72)	(79,59)	(45,29)	b	c	$P_{los}[b,d]$ is updated from (58,72) to (79,59); $pred[b,d]$ is updated from b to c
17	c	d	a	(50,40)	(79,59)	(36,28)	d	c	
18	c	d	b	(67,55)	(79,59)	(67,55)	a	a	
19	d	a	b	(67,55)	(79,59)	(67,55)	a	a	
20	d	a	c	(79,59)	(79,59)	(79,59)	b	b	
21	d	b	a	(79,59)	(79,59)	(50,40)	c	d	
22	d	b	c	(79,59)	(79,59)	(79,59)	b	b	
23	d	c	a	(36,28)	(45,29)	(50,40)	c	d	
24	d	c	b	(67,55)	(45,29)	(67,55)	a	a	

3.9. Example 9

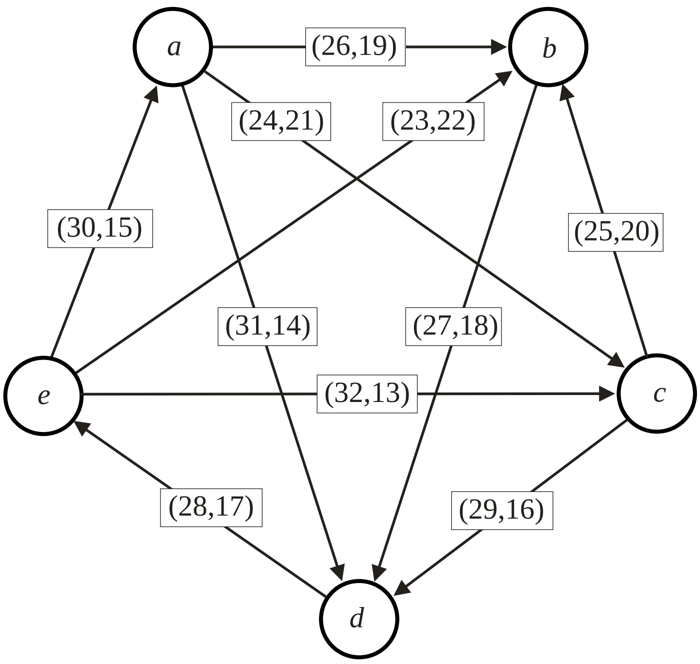
Example 9:

9	voters	$a \succ_v d \succ_v b \succ_v e \succ_v c$
6	voters	$b \succ_v c \succ_v a \succ_v d \succ_v e$
5	voters	$b \succ_v c \succ_v d \succ_v e \succ_v a$
2	voters	$c \succ_v d \succ_v b \succ_v e \succ_v a$
6	voters	$d \succ_v e \succ_v c \succ_v b \succ_v a$
14	voters	$e \succ_v a \succ_v c \succ_v b \succ_v d$
2	voters	$e \succ_v c \succ_v a \succ_v b \succ_v d$
1	voter	$e \succ_v d \succ_v a \succ_v c \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	26	24	31	15
$N[b,*]$	19	---	20	27	22
$N[c,*]$	21	25	---	29	13
$N[d,*]$	14	18	16	---	28
$N[e,*]$	30	23	32	17	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (26,19), *b*	*a*, (31,14), *d*, (28,17), *e*, (32,13), *c*	*a*, (31,14), *d*	*a*, (31,14), *d*, (28,17), *e*
from *b* ...	*b*, (27,18), *d*, (28,17), *e*, (30,15), *a*	---	*b*, (27,18), *d*, (28,17), *e*, (32,13), *c*	*b*, (27,18), *d*	*b*, (27,18), *d*, (28,17), *e*
from *c* ...	*c*, (29,16), *d*, (28,17), *e*, (30,15), *a*	*c*, (29,16), *d*, (28,17), *e*, (30,15), *a*, (26,19), *b*	---	*c*, (29,16), *d*	*c*, (29,16), *d*, (28,17), *e*
from *d* ...	*d*, (28,17), *e*, (30,15), *a*	*d*, (28,17), *e*, (30,15), *a*, (26,19), *b*	*d*, (28,17), *e*, (32,13), *c*	---	*d*, (28,17), *e*
from *e* ...	*e*, (30,15), *a*	*e*, (30,15), *a*, (26,19), *b*	*e*, (32,13), *c*	*e*, (30,15), *a*, (31,14), *d*	---

We get $\mathcal{O}$ = {*ad*, *ba*, *bc*, *bd*, *be*, *cd*, *ea*, *ec*, *ed*} and $\mathcal{S}$ = {*b*}.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 60$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(20,25)	(19,26)	(24,21)	b	a	
2	a	b	d	(27,18)	(19,26)	(31,14)	b	a	
3	a	b	e	(22,23)	(19,26)	(15,30)	b	a	
4	a	c	b	(25,20)	(21,24)	(26,19)	c	a	
5	a	c	d	(29,16)	(21,24)	(31,14)	c	a	
6	a	c	e	(13,32)	(21,24)	(15,30)	c	a	$P_D[c,e]$ is updated from (13,32) to (15,30); $pred[c,e]$ is updated from c to a
7	a	d	b	(18,27)	(14,31)	(26,19)	d	a	
8	a	d	c	(16,29)	(14,31)	(24,21)	d	a	
9	a	d	e	(28,17)	(14,31)	(15,30)	d	a	
10	a	e	b	(23,22)	(30,15)	(26,19)	e	a	$P_D[e,b]$ is updated from (23,22) to (26,19); $pred[e,b]$ is updated from e to a
11	a	e	c	(32,13)	(30,15)	(24,21)	e	a	
12	a	e	d	(17,28)	(30,15)	(31,14)	e	a	$P_D[e,d]$ is updated from (17,28) to (30,15); $pred[e,d]$ is updated from e to a
13	b	a	c	(24,21)	(26,19)	(20,25)	a	b	
14	b	a	d	(31,14)	(26,19)	(27,18)	a	b	
15	b	a	e	(15,30)	(26,19)	(22,23)	a	b	$P_D[a,e]$ is updated from (15,30) to (22,23); $pred[a,e]$ is updated from a to b
16	b	c	a	(21,24)	(25,20)	(19,26)	c	b	
17	b	c	d	(29,16)	(25,20)	(27,18)	c	b	
18	b	c	e	(15,30)	(25,20)	(22,23)	a	b	$P_D[c,e]$ is updated from (15,30) to (22,23); $pred[c,e]$ is updated from a to b
19	b	d	a	(14,31)	(18,27)	(19,26)	d	b	$P_D[d,a]$ is updated from (14,31) to (18,27); $pred[d,a]$ is updated from d to b
20	b	d	c	(16,29)	(18,27)	(20,25)	d	b	$P_D[d,c]$ is updated from (16,29) to (18,27); $pred[d,c]$ is updated from d to b
21	b	d	e	(28,17)	(18,27)	(22,23)	d	b	
22	b	e	a	(30,15)	(26,19)	(19,26)	e	b	
23	b	e	c	(32,13)	(26,19)	(20,25)	e	b	
24	b	e	d	(30,15)	(26,19)	(27,18)	a	b	
25	c	a	b	(26,19)	(24,21)	(25,20)	a	c	
26	c	a	d	(31,14)	(24,21)	(29,16)	a	c	
27	c	a	e	(22,23)	(24,21)	(22,23)	b	b	
28	c	b	a	(19,26)	(20,25)	(21,24)	b	c	$P_D[b,a]$ is updated from (19,26) to (20,25); $pred[b,a]$ is updated from b to c
29	c	b	d	(27,18)	(20,25)	(29,16)	b	c	
30	c	b	e	(22,23)	(20,25)	(22,23)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	*c*	*d*	*a*	(18,27)	(18,27)	(21,24)	*b*	*c*	
32	*c*	*d*	*b*	(18,27)	(18,27)	(25,20)	*d*	*c*	
33	*c*	*d*	*e*	(28,17)	(18,27)	(22,23)	*d*	*b*	
34	*c*	*e*	*a*	(30,15)	(32,13)	(21,24)	*e*	*c*	
35	*c*	*e*	*b*	(26,19)	(32,13)	(25,20)	*a*	*c*	
36	*c*	*e*	*d*	(30,15)	(32,13)	(29,16)	*a*	*c*	
37	*d*	*a*	*b*	(26,19)	(31,14)	(18,27)	*a*	*d*	
38	*d*	*a*	*c*	(24,21)	(31,14)	(18,27)	*a*	*b*	
39	*d*	*a*	*e*	(22,23)	(31,14)	(28,17)	*b*	*d*	$P_D[a,e]$ is updated from (22,23) to (28,17); $pred[a,e]$ is updated from *b* to *d*
40	*d*	*b*	*a*	(20,25)	(27,18)	(18,27)	*c*	*b*	
41	*d*	*b*	*c*	(20,25)	(27,18)	(18,27)	*b*	*b*	
42	*d*	*b*	*e*	(22,23)	(27,18)	(28,17)	*b*	*d*	$P_D[b,e]$ is updated from (22,23) to (27,18); $pred[b,e]$ is updated from *b* to *d*
43	*d*	*c*	*a*	(21,24)	(29,16)	(18,27)	*c*	*b*	
44	*d*	*c*	*b*	(25,20)	(29,16)	(18,27)	*c*	*d*	
45	*d*	*c*	*e*	(22,23)	(29,16)	(28,17)	*b*	*d*	$P_D[c,e]$ is updated from (22,23) to (28,17); $pred[c,e]$ is updated from *b* to *d*
46	*d*	*e*	*a*	(30,15)	(30,15)	(18,27)	*e*	*b*	
47	*d*	*e*	*b*	(26,19)	(30,15)	(18,27)	*a*	*d*	
48	*d*	*e*	*c*	(32,13)	(30,15)	(18,27)	*e*	*b*	
49	*e*	*a*	*b*	(26,19)	(28,17)	(26,19)	*a*	*a*	
50	*e*	*a*	*c*	(24,21)	(28,17)	(32,13)	*a*	*e*	$P_D[a,c]$ is updated from (24,21) to (28,17); $pred[a,c]$ is updated from *a* to *e*
51	*e*	*a*	*d*	(31,14)	(28,17)	(30,15)	*a*	*a*	
52	*e*	*b*	*a*	(20,25)	(27,18)	(30,15)	*c*	*e*	$P_D[b,a]$ is updated from (20,25) to (27,18); $pred[b,a]$ is updated from *c* to *e*
53	*e*	*b*	*c*	(20,25)	(27,18)	(32,13)	*b*	*e*	$P_D[b,c]$ is updated from (20,25) to (27,18); $pred[b,c]$ is updated from *b* to *e*
54	*e*	*b*	*d*	(27,18)	(27,18)	(30,15)	*b*	*a*	
55	*e*	*c*	*a*	(21,24)	(28,17)	(30,15)	*c*	*e*	$P_D[c,a]$ is updated from (21,24) to (28,17); $pred[c,a]$ is updated from *c* to *e*
56	*e*	*c*	*b*	(25,20)	(28,17)	(26,19)	*c*	*a*	$P_D[c,b]$ is updated from (25,20) to (26,19); $pred[c,b]$ is updated from *c* to *a*
57	*e*	*c*	*d*	(29,16)	(28,17)	(30,15)	*c*	*a*	
58	*e*	*d*	*a*	(18,27)	(28,17)	(30,15)	*b*	*e*	$P_D[d,a]$ is updated from (18,27) to (28,17); $pred[d,a]$ is updated from *b* to *e*
59	*e*	*d*	*b*	(18,27)	(28,17)	(26,19)	*d*	*a*	$P_D[d,b]$ is updated from (18,27) to (26,19); $pred[d,b]$ is updated from *d* to *a*
60	*e*	*d*	*c*	(18,27)	(28,17)	(32,13)	*b*	*e*	$P_D[d,c]$ is updated from (18,27) to (28,17); $pred[d,c]$ is updated from *b* to *e*

3.10. Example 10

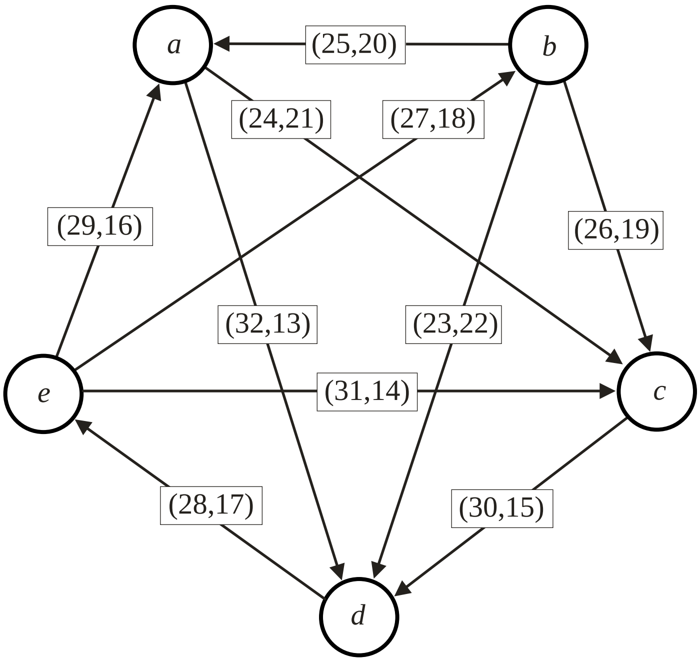
Example 10:

9	voters	$a \succ_v d \succ_v b \succ_v e \succ_v c$
1	voter	$b \succ_v a \succ_v c \succ_v e \succ_v d$
6	voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
2	voters	$c \succ_v d \succ_v b \succ_v e \succ_v a$
5	voters	$c \succ_v d \succ_v e \succ_v a \succ_v b$
6	voters	$d \succ_v e \succ_v c \succ_v a \succ_v b$
14	voters	$e \succ_v b \succ_v a \succ_v c \succ_v d$
2	voters	$e \succ_v b \succ_v c \succ_v a \succ_v d$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	20	24	32	16
$N[b,*]$	25	---	26	23	18
$N[c,*]$	21	19	---	30	14
$N[d,*]$	13	22	15	---	28
$N[e,*]$	29	27	31	17	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (32,13), *d*, (28,17), *e*, <u>(27,18)</u>, *b*	*a*, (32,13), *d*, <u>(28,17)</u>, *e*, (31,14), *c*	*a*, <u>(32,13)</u>, *d*	*a*, (32,13), *d*, <u>(28,17)</u>, *e*
from *b* ...	*b*, <u>(26,19)</u>, *c*, (30,15), *d*, (28,17), *e*, (29,16), *a*	---	*b*, <u>(26,19)</u>, *c*	*b*, <u>(26,19)</u>, *c*, (30,15), *d*	*b*, <u>(26,19)</u>, *c*, (30,15), *d*, (28,17), *e*
from *c* ...	*c*, (30,15), *d*, <u>(28,17)</u>, *e*, (29,16), *a*	*c*, (30,15), *d*, (28,17), *e*, <u>(27,18)</u>, *b*	---	*c*, <u>(30,15)</u>, *d*	*c*, (30,15), *d*, <u>(28,17)</u>, *e*
from *d* ...	*d*, <u>(28,17)</u>, *e*, (29,16), *a*	*d*, (28,17), *e*, <u>(27,18)</u>, *b*	*d*, <u>(28,17)</u>, *e*, (31,14), *c*	---	*d*, <u>(28,17)</u>, *e*
from *e* ...	*e*, <u>(29,16)</u>, *a*	*e*, <u>(27,18)</u>, *b*	*e*, <u>(31,14)</u>, *c*	*e*, (31,14), *c*, <u>(30,15)</u>, *d*	---

We get $\mathcal{O}$ = {*ab*, *ad*, *cb*, *cd*, *db*, *ea*, *eb*, *ec*, *ed*} and $\mathcal{S}$ = {*e*}.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 60$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(26,19)	(25,20)	(24,21)	b	a	
2	a	b	d	(23,22)	(25,20)	(32,13)	b	a	$P_D[b,d]$ is updated from (23,22) to (25,20); $pred[b,d]$ is updated from b to a
3	a	b	e	(18,27)	(25,20)	(16,29)	b	a	
4	a	c	b	(19,26)	(21,24)	(20,25)	c	a	$P_D[c,b]$ is updated from (19,26) to (20,25); $pred[c,b]$ is updated from c to a
5	a	c	d	(30,15)	(21,24)	(32,13)	c	a	
6	a	c	e	(14,31)	(21,24)	(16,29)	c	a	$P_D[c,e]$ is updated from (14,31) to (16,29); $pred[c,e]$ is updated from c to a
7	a	d	b	(22,23)	(13,32)	(20,25)	d	a	
8	a	d	c	(15,30)	(13,32)	(24,21)	d	a	
9	a	d	e	(28,17)	(13,32)	(16,29)	d	a	
10	a	e	b	(27,18)	(29,16)	(20,25)	e	a	
11	a	e	c	(31,14)	(29,16)	(24,21)	e	a	
12	a	e	d	(17,28)	(29,16)	(32,13)	e	a	$P_D[e,d]$ is updated from (17,28) to (29,16); $pred[e,d]$ is updated from e to a
13	b	a	c	(24,21)	(20,25)	(26,19)	a	b	
14	b	a	d	(32,13)	(20,25)	(25,20)	a	a	
15	b	a	e	(16,29)	(20,25)	(18,27)	a	b	$P_D[a,e]$ is updated from (16,29) to (18,27); $pred[a,e]$ is updated from a to b
16	b	c	a	(21,24)	(20,25)	(25,20)	c	b	
17	b	c	d	(30,15)	(20,25)	(25,20)	c	a	
18	b	c	e	(16,29)	(20,25)	(18,27)	a	b	$P_D[c,e]$ is updated from (16,29) to (18,27); $pred[c,e]$ is updated from a to b
19	b	d	a	(13,32)	(22,23)	(25,20)	d	b	$P_D[d,a]$ is updated from (13,32) to (22,23); $pred[d,a]$ is updated from d to b
20	b	d	c	(15,30)	(22,23)	(26,19)	d	b	$P_D[d,c]$ is updated from (15,30) to (22,23); $pred[d,c]$ is updated from d to b
21	b	d	e	(28,17)	(22,23)	(18,27)	d	b	
22	b	e	a	(29,16)	(27,18)	(25,20)	e	b	
23	b	e	c	(31,14)	(27,18)	(26,19)	e	b	
24	b	e	d	(29,16)	(27,18)	(25,20)	a	a	
25	c	a	b	(20,25)	(24,21)	(20,25)	a	a	
26	c	a	d	(32,13)	(24,21)	(30,15)	a	c	
27	c	a	e	(18,27)	(24,21)	(18,27)	b	b	
28	c	b	a	(25,20)	(26,19)	(21,24)	b	c	
29	c	b	d	(25,20)	(26,19)	(30,15)	a	c	$P_D[b,d]$ is updated from (25,20) to (26,19); $pred[b,d]$ is updated from a to c
30	c	b	e	(18,27)	(26,19)	(18,27)	b	b	

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(22,23)	(22,23)	(21,24)	b	c	
32	c	d	b	(22,23)	(22,23)	(20,25)	d	a	
33	c	d	e	(28,17)	(22,23)	(18,27)	d	b	
34	c	e	a	(29,16)	(31,14)	(21,24)	e	c	
35	c	e	b	(27,18)	(31,14)	(20,25)	e	a	
36	c	e	d	(29,16)	(31,14)	(30,15)	a	c	$P_D[e,d]$ is updated from (29,16) to (30,15); $pred[e,d]$ is updated from a to c
37	d	a	b	(20,25)	(32,13)	(22,23)	a	d	$P_D[a,b]$ is updated from (20,25) to (22,23); $pred[a,b]$ is updated from a to d
38	d	a	c	(24,21)	(32,13)	(22,23)	a	b	
39	d	a	e	(18,27)	(32,13)	(28,17)	b	d	$P_D[a,e]$ is updated from (18,27) to (28,17); $pred[a,e]$ is updated from b to d
40	d	b	a	(25,20)	(26,19)	(22,23)	b	b	
41	d	b	c	(26,19)	(26,19)	(22,23)	b	b	
42	d	b	e	(18,27)	(26,19)	(28,17)	b	d	$P_D[b,e]$ is updated from (18,27) to (26,19); $pred[b,e]$ is updated from b to d
43	d	c	a	(21,24)	(30,15)	(22,23)	c	b	$P_D[c,a]$ is updated from (21,24) to (22,23); $pred[c,a]$ is updated from c to b
44	d	c	b	(20,25)	(30,15)	(22,23)	a	d	$P_D[c,b]$ is updated from (20,25) to (22,23); $pred[c,b]$ is updated from a to d
45	d	c	e	(18,27)	(30,15)	(28,17)	b	d	$P_D[c,e]$ is updated from (18,27) to (28,17); $pred[c,e]$ is updated from b to d
46	d	e	a	(29,16)	(30,15)	(22,23)	e	b	
47	d	e	b	(27,18)	(30,15)	(22,23)	e	d	
48	d	e	c	(31,14)	(30,15)	(22,23)	e	b	
49	e	a	b	(22,23)	(28,17)	(27,18)	d	e	$P_D[a,b]$ is updated from (22,23) to (27,18); $pred[a,b]$ is updated from d to e
50	e	a	c	(24,21)	(28,17)	(31,14)	a	e	$P_D[a,c]$ is updated from (24,21) to (28,17); $pred[a,c]$ is updated from a to e
51	e	a	d	(32,13)	(28,17)	(30,15)	a	c	
52	e	b	a	(25,20)	(26,19)	(29,16)	b	e	$P_D[b,a]$ is updated from (25,20) to (26,19); $pred[b,a]$ is updated from b to e
53	e	b	c	(26,19)	(26,19)	(31,14)	b	e	
54	e	b	d	(26,19)	(26,19)	(30,15)	c	c	
55	e	c	a	(22,23)	(28,17)	(29,16)	b	e	$P_D[c,a]$ is updated from (22,23) to (28,17); $pred[c,a]$ is updated from b to e
56	e	c	b	(22,23)	(28,17)	(27,18)	d	e	$P_D[c,b]$ is updated from (22,23) to (27,18); $pred[c,b]$ is updated from d to e
57	e	c	d	(30,15)	(28,17)	(30,15)	c	c	
58	e	d	a	(22,23)	(28,17)	(29,16)	b	e	$P_D[d,a]$ is updated from (22,23) to (28,17); $pred[d,a]$ is updated from b to e
59	e	d	b	(22,23)	(28,17)	(27,18)	d	e	$P_D[d,b]$ is updated from (22,23) to (27,18); $pred[d,b]$ is updated from d to e
60	e	d	c	(22,23)	(28,17)	(31,14)	b	e	$P_D[d,c]$ is updated from (22,23) to (28,17); $pred[d,c]$ is updated from b to e

3.11. Example 11

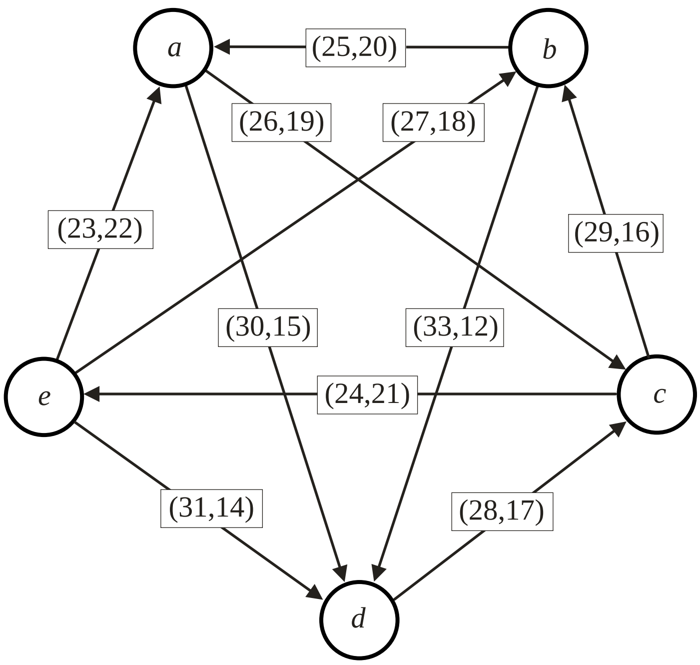
Example 11:

5	voters	$a \succ_v c \succ_v b \succ_v e \succ_v d$
5	voters	$a \succ_v d \succ_v e \succ_v c \succ_v b$
8	voters	$b \succ_v e \succ_v d \succ_v a \succ_v c$
3	voters	$c \succ_v a \succ_v b \succ_v e \succ_v d$
7	voters	$c \succ_v a \succ_v e \succ_v b \succ_v d$
2	voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
7	voters	$d \succ_v c \succ_v e \succ_v b \succ_v a$
8	voters	$e \succ_v b \succ_v a \succ_v d \succ_v c$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$	$N[*,e]$
$N[a,*]$	---	20	26	30	22
$N[b,*]$	25	---	16	33	18
$N[c,*]$	19	29	---	17	24
$N[d,*]$	15	12	28	---	14
$N[e,*]$	23	27	21	31	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, (30,15), *d*, (28,17), *c*, (29,16), *b*	*a*, (30,15), *d*, (28,17), *c*	*a*, (30,15), *d*	*a*, (30,15), *d*, (28,17), *c*, (24,21), *e*
from *b* ...	*b*, (25,20), *a*	---	*b*, (33,12), *d*, (28,17), *c*	*b*, (33,12), *d*	*b*, (33,12), *d*, (28,17), *c*, (24,21), *e*
from *c* ...	*c*, (29,16), *b*, (25,20), *a*	*c*, (29,16), *b*	---	*c*, (29,16), *b*, (33,12), *d*	*c*, (24,21), *e*
from *d* ...	*d*, (28,17), *c*, (29,16), *b*, (25,20), *a*	*d*, (28,17), *c*, (29,16), *b*	*d*, (28,17), *c*	---	*d*, (28,17), *c*, (24,21), *e*
from *e* ...	*e*, (31,14), *d*, (28,17), *c*, (29,16), *b*, (25,20), *a*	*e*, (31,14), *d*, (28,17), *c*, (29,16), *b*	*e*, (31,14), *d*, (28,17), *c*	*e*, (31,14), *d*	---

We get $\mathcal{O} = \{ab, ac, ad, bd, cb, cd, ea, eb, ec, ed\}$ and $\mathcal{S} = \{e\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 60$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(16,29)	(25,20)	(26,19)	b	a	$P_D[b,c]$ is updated from (16,29) to (25,20); $pred[b,c]$ is updated from b to a
2	a	b	d	(33,12)	(25,20)	(30,15)	b	a	
3	a	b	e	(18,27)	(25,20)	(22,23)	b	a	$P_D[b,e]$ is updated from (18,27) to (22,23); $pred[b,e]$ is updated from b to a
4	a	c	b	(29,16)	(19,26)	(20,25)	c	a	
5	a	c	d	(17,28)	(19,26)	(30,15)	c	a	$P_D[c,d]$ is updated from (17,28) to (19,26); $pred[c,d]$ is updated from c to a
6	a	c	e	(24,21)	(19,26)	(22,23)	c	a	
7	a	d	b	(12,33)	(15,30)	(20,25)	d	a	$P_D[d,b]$ is updated from (12,33) to (15,30); $pred[d,b]$ is updated from d to a
8	a	d	c	(28,17)	(15,30)	(26,19)	d	a	
9	a	d	e	(14,31)	(15,30)	(22,23)	d	a	$P_D[d,e]$ is updated from (14,31) to (15,30); $pred[d,e]$ is updated from d to a
10	a	e	b	(27,18)	(23,22)	(20,25)	e	a	
11	a	e	c	(21,24)	(23,22)	(26,19)	e	a	$P_D[e,c]$ is updated from (21,24) to (23,22); $pred[e,c]$ is updated from e to a
12	a	e	d	(31,14)	(23,22)	(30,15)	e	a	
13	b	a	c	(26,19)	(20,25)	(25,20)	a	a	
14	b	a	d	(30,15)	(20,25)	(33,12)	a	b	
15	b	a	e	(22,23)	(20,25)	(22,23)	a	a	
16	b	c	a	(19,26)	(29,16)	(25,20)	c	b	$P_D[c,a]$ is updated from (19,26) to (25,20); $pred[c,a]$ is updated from c to b
17	b	c	d	(19,26)	(29,16)	(33,12)	a	b	$P_D[c,d]$ is updated from (19,26) to (29,16); $pred[c,d]$ is updated from a to b
18	b	c	e	(24,21)	(29,16)	(22,23)	c	a	
19	b	d	a	(15,30)	(15,30)	(25,20)	d	b	
20	b	d	c	(28,17)	(15,30)	(25,20)	d	a	
21	b	d	e	(15,30)	(15,30)	(22,23)	a	a	
22	b	e	a	(23,22)	(27,18)	(25,20)	e	b	$P_D[e,a]$ is updated from (23,22) to (25,20); $pred[e,a]$ is updated from e to b
23	b	e	c	(23,22)	(27,18)	(25,20)	a	a	$P_D[e,c]$ is updated from (23,22) to (25,20)
24	b	e	d	(31,14)	(27,18)	(33,12)	e	b	
25	c	a	b	(20,25)	(26,19)	(29,16)	a	c	$P_D[a,b]$ is updated from (20,25) to (26,19); $pred[a,b]$ is updated from a to c
26	c	a	d	(30,15)	(26,19)	(29,16)	a	b	
27	c	a	e	(22,23)	(26,19)	(24,21)	a	c	$P_D[a,e]$ is updated from (22,23) to (24,21); $pred[a,e]$ is updated from a to c
28	c	b	a	(25,20)	(25,20)	(25,20)	b	b	
29	c	b	d	(33,12)	(25,20)	(29,16)	b	b	
30	c	b	e	(22,23)	(25,20)	(24,21)	a	c	$P_D[b,e]$ is updated from (22,23) to (24,21); $pred[b,e]$ is updated from a to c

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
31	c	d	a	(15,30)	(28,17)	(25,20)	d	b	$P_D[d,a]$ is updated from (15,30) to (25,20); $pred[d,a]$ is updated from d to b
32	c	d	b	(15,30)	(28,17)	(29,16)	a	c	$P_D[d,b]$ is updated from (15,30) to (28,17); $pred[d,b]$ is updated from a to c
33	c	d	e	(15,30)	(28,17)	(24,21)	a	c	$P_D[d,e]$ is updated from (15,30) to (24,21); $pred[d,e]$ is updated from a to c
34	c	e	a	(25,20)	(25,20)	(25,20)	b	b	
35	c	e	b	(27,18)	(25,20)	(29,16)	e	c	
36	c	e	d	(31,14)	(25,20)	(29,16)	e	b	
37	d	a	b	(26,19)	(30,15)	(28,17)	c	c	$P_D[a,b]$ is updated from (26,19) to (28,17)
38	d	a	c	(26,19)	(30,15)	(28,17)	a	d	$P_D[a,c]$ is updated from (26,19) to (28,17); $pred[a,c]$ is updated from a to d
39	d	a	e	(24,21)	(30,15)	(24,21)	c	c	
40	d	b	a	(25,20)	(33,12)	(25,20)	b	b	
41	d	b	c	(25,20)	(33,12)	(28,17)	a	d	$P_D[b,c]$ is updated from (25,20) to (28,17); $pred[b,c]$ is updated from a to d
42	d	b	e	(24,21)	(33,12)	(24,21)	c	c	
43	d	c	a	(25,20)	(29,16)	(25,20)	b	b	
44	d	c	b	(29,16)	(29,16)	(28,17)	c	c	
45	d	c	e	(24,21)	(29,16)	(24,21)	c	c	
46	d	e	a	(25,20)	(31,14)	(25,20)	b	b	
47	d	e	b	(27,18)	(31,14)	(28,17)	e	c	$P_D[e,b]$ is updated from (27,18) to (28,17); $pred[e,b]$ is updated from e to c
48	d	e	c	(25,20)	(31,14)	(28,17)	a	d	$P_D[e,c]$ is updated from (25,20) to (28,17); $pred[e,c]$ is updated from a to d
49	e	a	b	(28,17)	(24,21)	(28,17)	c	c	
50	e	a	c	(28,17)	(24,21)	(28,17)	d	d	
51	e	a	d	(30,15)	(24,21)	(31,14)	a	e	
52	e	b	a	(25,20)	(24,21)	(25,20)	b	b	
53	e	b	c	(28,17)	(24,21)	(28,17)	d	d	
54	e	b	d	(33,12)	(24,21)	(31,14)	b	e	
55	e	c	a	(25,20)	(24,21)	(25,20)	b	b	
56	e	c	b	(29,16)	(24,21)	(28,17)	c	c	
57	e	c	d	(29,16)	(24,21)	(31,14)	b	e	
58	e	d	a	(25,20)	(24,21)	(25,20)	b	b	
59	e	d	b	(28,17)	(24,21)	(28,17)	c	c	
60	e	d	c	(28,17)	(24,21)	(28,17)	d	d	

3.12. Example 12

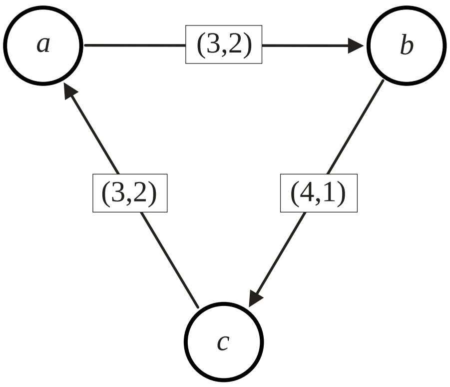
Example 12:

2	voters	$a \succ_v b \succ_v c$
2	voters	$b \succ_v c \succ_v a$
1	voter	$c \succ_v a \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$
$N[a,*]$	---	3	2
$N[b,*]$	2	---	4
$N[c,*]$	3	1	---

The corresponding digraph looks as follows:



The following table lists the strongest paths. The critical links of the strongest paths are <u>underlined</u>:

	... to a	... to b	... to c
from a ...	---	a, <u>(3,2)</u>, b	a, <u>(3,2)</u>, b, (4,1), c
from b ...	b, (4,1), c, <u>(3,2)</u>, a	---	b, <u>(4,1)</u>, c
from c ...	c, <u>(3,2)</u>, a	c, <u>(3,2)</u>, a, <u>(3,2)</u>, b	---

We get $\mathcal{O} = \{bc\}$ and $\mathcal{S} = \{a, b\}$.

Suppose, the strongest paths are calculated with the Floyd-Warshall algorithm, as defined in section 2.3. Then the following table documents the $C \cdot (C–1) \cdot (C–2) = 6$ steps of the Floyd-Warshall algorithm.

We start with

- $P_D[i,j] := (N[i,j],N[j,i])$ for all $i \in A$ and $j \in A \setminus \{i\}$.
- $pred[i,j] := i$ for all $i \in A$ and $j \in A \setminus \{i\}$.

	i	j	k	$P_D[j,k]$	$P_D[j,i]$	$P_D[i,k]$	$pred[j,k]$	$pred[i,k]$	result
1	a	b	c	(4,1)	(2,3)	(2,3)	b	a	
2	a	c	b	(1,4)	(3,2)	(3,2)	c	a	$P_D[c,b]$ is updated from (1,4) to (3,2); $pred[c,b]$ is updated from c to a
3	b	a	c	(2,3)	(3,2)	(4,1)	a	b	$P_D[a,c]$ is updated from (2,3) to (3,2); $pred[a,c]$ is updated from a to b
4	b	c	a	(3,2)	(3,2)	(2,3)	c	b	
5	c	a	b	(3,2)	(3,2)	(3,2)	a	a	
6	c	b	a	(2,3)	(4,1)	(3,2)	b	c	$P_D[b,a]$ is updated from (2,3) to (3,2); $pred[b,a]$ is updated from b to c

4. Analysis of the Schulze Method

4.1. Transitivity

In this section, we will prove that the binary relation $\mathcal{O}$, as defined in (2.2.1), is *transitive*. This means: If $ab \in \mathcal{O}$ and $bc \in \mathcal{O}$, then $ac \in \mathcal{O}$. This guarantees that the set $\mathcal{S}$ of potential winners, as defined in (2.2.2), is non-empty. When we interpret the Schulze method as a method to find a set $\mathcal{S}$ of potential winners, rather than a method to generate a binary relation $\mathcal{O}$, then the proof of the transitivity of $\mathcal{O}$ is an essential part of the proof that the Schulze method is well defined.

Definition:

An election method satisfies *transitivity* if the following holds for all $a,b,c \in A$:

Suppose:

(4.1.1) $ab \in \mathcal{O}$.

(4.1.2) $bc \in \mathcal{O}$.

Then:

(4.1.3) $ac \in \mathcal{O}$.

Claim:

The binary relation $\mathcal{O}$, as defined in (2.2.1), is transitive.

Proof:

With (4.1.1), we get

(4.1.4) $P_D[a,b] \succ_D P_D[b,a]$.

With (4.1.2), we get

(4.1.5) $P_D[b,c] \succ_D P_D[c,b]$.

With (2.2.5), we get

(4.1.6) $\min_D \{ P_D[a,b], P_D[b,c] \} \precsim_D P_D[a,c]$.

(4.1.7) $\min_D \{ P_D[b,c], P_D[c,a] \} \precsim_D P_D[b,a]$.

(4.1.8) $\min_D \{ P_D[c,a], P_D[a,b] \} \precsim_D P_D[c,b]$.

<u>Case 1:</u> Suppose

(4.1.9a) $P_D[a,b] \succsim_D P_D[b,c]$.

Combining (4.1.5) and (4.1.9a) gives

(4.1.10a) $P_D[a,b] \succ_D P_D[c,b]$.

Combining (4.1.8) and (4.1.10a) gives

(4.1.11a) $P_D[c,a] \precsim_D P_D[c,b]$.

Combining (4.1.6) and (4.1.9a) gives

(4.1.12a) $P_D[b,c] \precsim_D P_D[a,c]$.

Combining (4.1.11a), (4.1.5), and (4.1.12a) gives

(4.1.13a) $P_D[c,a] \precsim_D P_D[c,b] \prec_D P_D[b,c] \precsim_D P_D[a,c]$.

With (4.1.13a), we get (4.1.3).

<u>Case 2:</u> Suppose

(4.1.9b) $P_D[a,b] \prec_D P_D[b,c]$.

Combining (4.1.4) and (4.1.9b) gives

(4.1.10b) $P_D[b,a] \prec_D P_D[b,c]$.

Combining (4.1.7) and (4.1.10b) gives

(4.1.11b) $P_D[c,a] \precsim_D P_D[b,a]$.

Combining (4.1.6) and (4.1.9b) gives

(4.1.12b) $P_D[a,b] \precsim_D P_D[a,c]$.

Combining (4.1.11b), (4.1.4), and (4.1.12b) gives

(4.1.13b) $P_D[c,a] \precsim_D P_D[b,a] \prec_D P_D[a,b] \precsim_D P_D[a,c]$.

With (4.1.13b), we get (4.1.3). □

The proof, that the Schulze method is transitive, has first been published by Schulze (1998).

The following corollary says that the set $\mathcal{S}$ of potential winners, as defined in (2.2.2), is non-empty.

Corollary (4.1.14):

For the Schulze method, as defined in section 2.2, we get

(4.1.14) $\forall\ b \notin \mathcal{S}\ \exists\ a \in \mathcal{S}: ab \in \mathcal{O}$.

Proof of corollary (4.1.14):

As $b \notin \mathcal{S}$, there must be a $c(1) \in A$ with $c(1),b \in \mathcal{O}$.

If $c(1) \in \mathcal{S}$, then the corollary is proven. If $c(1) \notin \mathcal{S}$, then there must be a $c(2) \in A$ with $c(2),c(1) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(2),b \in \mathcal{O}$ and $c(2) \notin \{b, c(1)\}$.

We now proceed as follows: If $c(i) \in \mathcal{S}$, then the corollary is proven. If $c(i) \notin \mathcal{S}$, then there must be a $c(i+1) \in A$ with $c(i+1),c(i) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(i+1),b \in \mathcal{O}$ and $c(i+1) \notin \{b, c(1), ..., c(i)\}$.

We proceed until $c(i) \in \mathcal{S}$ for some $i \in \mathbb{N}$. Such an $i \in \mathbb{N}$ exists because A is finite. □

The following corollary says that alternative $a \in A$ is the unique winner if and only if alternative a disqualifies every other alternative $b \in A \setminus \{a\}$.

Corollary (4.1.15):

For the Schulze method, as defined in section 2.2, we get

(4.1.15) $\mathcal{S} = \{a\} \Leftrightarrow ab \in \mathcal{O}\ \forall\ b \in A \setminus \{a\}$.

Proof of corollary (4.1.15):

$\Leftarrow$ If $ab \in \mathcal{O}\ \forall\ b \in A \setminus \{a\}$, then $a \in A$ disqualifies every $b \in A \setminus \{a\}$ according to (2.2.2). Therefore, we get $\mathcal{S} = \{a\}$.

$\Rightarrow$ With (4.1.14) and $\mathcal{S} = \{a\}$, we get

(4.1.16) $\forall\ b \notin \mathcal{S}: ab \in \mathcal{O}$.

With $\mathcal{S} = \{a\}$, we get

(4.1.17) $b \notin \mathcal{S} \Leftrightarrow b \in A \setminus \{a\}$.

With (4.1.16) and (4.1.17), we get

(4.1.18) $\forall\ b \in A \setminus \{a\}: ab \in \mathcal{O}$. □

In example 2 (section 3.2), we have $ba \notin \mathcal{O}$ and $ac \notin \mathcal{O}$ and $bc \in \mathcal{O}$. This shows that the Schulze relation, as defined in (2.2.1), is not necessarily negatively transitive.

4.2. Resolvability

Resolvability basically says that usually there is a unique winner $\mathcal{S} = \{a\}$. There are two different versions of the resolvability criterion. We will prove that the Schulze method, as defined in section 2.2, satisfies both.

4.2.1. Formulation #1

Definition:

An election method satisfies the first version of the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique winner tends to zero as the number of voters in the profile tends to infinity.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the resolvability criterion.

Proof (overview):

Suppose $(x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, according to (2.1.1), there is a $v_1 \in \mathbb{N}_0$ such that for all $w_1 \in \mathbb{N}_0$:

1. $w_1 < v_1 \Rightarrow (x_1,x_2) \succ_D (w_1,y_2)$.

2. $w_1 > v_1 \Rightarrow (x_1,x_2) \prec_D (w_1,y_2)$.

When the number of voters tends to infinity (i.e. when x_1, x_2, y_1, and y_2 become large), then the proportion of profiles, where the condition “$y_1 = v_1$” happens to be satisfied, tends to zero. Therefore, when the number of voters tends to infinity, then the proportion of profiles, where two links *ef* and *gh* happen to have equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$, tends to zero.

Therefore, we will prove that, unless there are links *ef* and *gh* of equivalent strengths, there is always a unique winner. We will prove this by showing that, when we simultaneously presume (a) that there is more than one potential winner and (b) that there are no links *ef* and *gh* of equivalent strengths, then we necessarily get to a contradiction.

Proof (details):

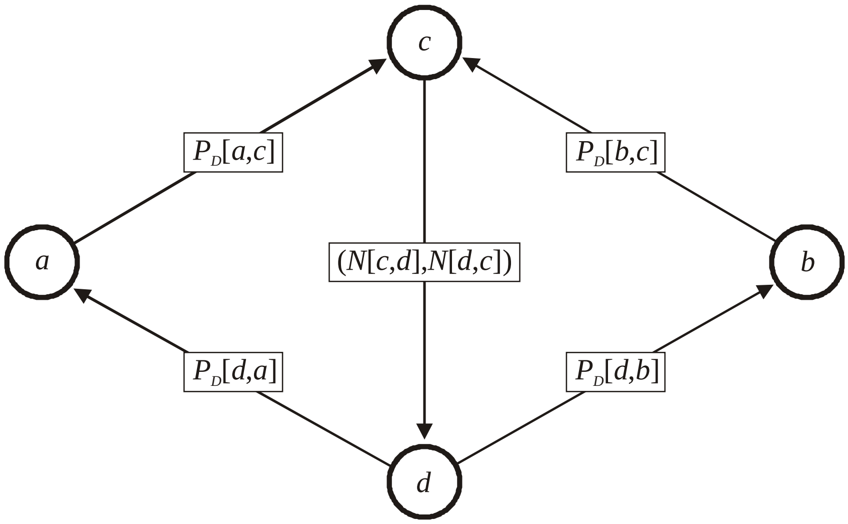
Suppose that there is more than one potential winner. Suppose alternative $a \in A$ and alternative $b \in A$ are potential winners. Then

(4.2.1.1) $\forall\, i \in A \setminus \{a\}: P_D[a,i] \succsim_D P_D[i,a]$.

(4.2.1.2) $\forall\, j \in A \setminus \{b\}: P_D[b,j] \succsim_D P_D[j,b]$.

(4.2.1.3) $P_D[a,b] \approx_D P_D[b,a]$.

Suppose there are no links *ef* and *gh* of equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$. Then $P_D[a,b] \approx_D P_D[b,a]$ means that the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *cd*. Therefore, the strongest paths have the following structure:



As *cd* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(4.2.1.4) $\quad P_D[a,d] \approx_D P_D[a,b]$.

(4.2.1.5) $\quad P_D[d,b] \succ_D P_D[a,b]$.

As *cd* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(4.2.1.6) $\quad P_D[b,d] \approx_D P_D[b,a]$.

(4.2.1.7) $\quad P_D[d,a] \succ_D P_D[b,a]$.

With (4.2.1.7), (4.2.1.3), and (4.2.1.4), we get

(4.2.1.8) $\quad P_D[d,a] \succ_D P_D[b,a] \approx_D P_D[a,b] \approx_D P_D[a,d]$.

But (4.2.1.8) contradicts (4.2.1.1).

Similarly, with (4.2.1.5), (4.2.1.3), and (4.2.1.6), we get

(4.2.1.9) $\quad P_D[d,b] \succ_D P_D[a,b] \approx_D P_D[b,a] \approx_D P_D[b,d]$.

But (4.2.1.9) contradicts (4.2.1.2). □

4.2.2. Formulation #2

The second version of the *resolvability criterion* says that, when there is more than one potential winner, then, for every alternative $a \in \mathcal{S}$, it is sufficient to add a single ballot w so that alternative a becomes the unique winner.

Definition:

An election method satisfies the second version of the *resolvability criterion* if the following holds:

$\forall\ a \in \mathcal{S}^{\text{old}}$: It is possible to construct a strict weak order w with the following two properties:

(4.2.2.1) $\forall\ f \in A \setminus \{a\}$: $a \succ_w f$.

(4.2.2.2) $\mathcal{S}^{\text{new}} = \{a\}$ for $V^{\text{new}} := V^{\text{old}} + \{w\}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the resolvability criterion.

Proof:

Suppose $a \in \mathcal{S}^{\text{old}}$. Then we get

(4.2.2.3) $\forall\ b \in A \setminus \{a\}$: $P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{old}}[b,a]$.

Suppose $pred^{\text{old}}[x,y]$ is the predecessor of alternative y in the strongest path from alternative $x \in A$ to alternative $y \in A \setminus \{x\}$, as calculated in section 2.3.

Suppose the strict weak order w is chosen as follows:

(4.2.2.4) $\forall\ f \in A \setminus \{a\}$: $pred^{\text{old}}[a,f] \succ_w f$.

(4.2.2.5) $\forall\ e,f \in A \setminus \{a\}$: $(\ P_D^{\text{old}}[e,a] \succ_D P_D^{\text{old}}[f,a] \Rightarrow e \succ_w f\ )$.

With (4.2.2.4), we get (4.2.2.1).

We will prove the following three claims:

Claim #1: It is not possible that (4.2.2.4) requires $e \succ_w f$ and that simultaneously (4.2.2.5) requires $f \succ_w e$.

Claim #2: $\forall\ g \in A \setminus \{a\}$: $P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.

Claim #3: $\forall\ g \in A \setminus \{a\}$: $P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g]$.

With claim #2 and claim #3, we get

$P_D^{\text{new}}[a,g] \succ_D P_D^{\text{new}}[g,a]$ for all $g \in A \setminus \{a\}$

so that $ag \in \mathcal{O}^{\text{new}}$ for all $g \in A \setminus \{a\}$

so that $\mathcal{S}^{\text{new}} = \{a\}$.

Proof of claim #1:

Suppose $e,f \in A \setminus \{a\}$. With (2.2.3), we get

(4.2.2.6) $\quad P_D^{\text{old}}[e,f] \succsim_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

With (2.2.5), we get

(4.2.2.7) $\quad \min_D \{ P_D^{\text{old}}[e,f], P_D^{\text{old}}[f,a] \} \precsim_D P_D^{\text{old}}[e,a]$.

With (4.2.2.3), we get

(4.2.2.8) $\quad P_D^{\text{old}}[a,f] \succsim_D P_D^{\text{old}}[f,a]$.

Suppose (4.2.2.4) requires $e \succ_w f$. Then $e = pred^{\text{old}}[a,f]$. Therefore, the link ef was in the strongest path from alternative a to alternative f. Thus, we get

(4.2.2.9) $\quad P_D^{\text{old}}[a,f] \precsim_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

Suppose (4.2.2.5) requires $f \succ_w e$. Then

(4.2.2.10) $\quad P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

With (4.2.2.6), (4.2.2.9), (4.2.2.8), and (4.2.2.10), we get

(4.2.2.11) $\quad P_D^{\text{old}}[e,f] \succsim_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \succsim_D P_D^{\text{old}}[a,f] \succsim_D P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

But (4.2.2.10) and (4.2.2.11) together contradict (4.2.2.7).

Proof of claim #2:

With (2.1.1) and (4.2.2.4), we get: The strength of each link of the strongest paths from alternative a to each other alternative $g \in A \setminus \{a\}$ is increased. Therefore

(4.2.2.12) $\quad \forall\, g \in A \setminus \{a\}: P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.

Proof of claim #3:

Suppose $g \in A \setminus \{a\}$. Suppose

(4.2.2.13) $\quad \Upsilon(g) := (\{a\} \cup \{ h \in A \setminus \{a\} \mid P^{\text{old}}_{D}[h,a] \succ_{D} P^{\text{old}}_{D}[a,g] \})$.

With (4.2.2.3) and (4.2.2.13), we get

(4.2.2.14) $\quad g \notin \Upsilon(g)$ and $a \in \Upsilon(g)$

and, therefore, $\emptyset \neq \Upsilon(g) \subsetneq A$. Furthermore, we get

(4.2.2.15) $\quad \forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{old}}[i,j],N^{\text{old}}[j,i]) \precsim_{D} P^{\text{old}}_{D}[a,g]$.

Otherwise, there was a path from alternative i to alternative a via alternative j with a strength of more than $P^{\text{old}}_{D}[a,g]$. But (as $i \notin \Upsilon(g)$) this would contradict the definition of $\Upsilon(g)$.

With (4.2.2.5), (4.2.2.1), and (4.2.2.13), we get

(4.2.2.16) $\quad \forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $j \succ_{w} i$.

With (2.1.1) and (4.2.2.16), we get

(4.2.2.17) $\quad \forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_{D} (N^{\text{old}}[i,j],N^{\text{old}}[j,i])$.

With (4.2.2.15) and (4.2.2.17), we get

(4.2.2.18) $\quad \forall\, i \notin \Upsilon(g)\ \forall\, j \in \Upsilon(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_{D} P^{\text{old}}_{D}[a,g]$.

With (4.2.2.14) and (4.2.2.18), we get

(4.2.2.19) $\quad P^{\text{new}}_{D}[g,a] \prec_{D} P^{\text{old}}_{D}[a,g]$. □

The proof in section 4.2.2 has first been published by Schulze (2011). It immediately attracted attention, because it doesn’t only prove that there is a tie-breaking ballot w, it also shows how this tie-breaking ballot w can be calculated in a polynomial runtime. Parkes and Xia (2012) pointed to the fact that this proof can also be interpreted as saying that it is possible to calculate a voting strategy in a polynomial runtime. This observation by Parkes and Xia has been extended by Gaspers (2012), Menton (2013a, 2013b), J. Müller (2013), Reisch (2014), Schend (2015), and Hemaspaandra (2016). Surveys, that are including the Schulze method, on the complexity of calculating a voting strategy have been written by Durand (2015), Baumeister and Rothe (2016), Conitzer and Walsh (2016), and Faliszewski and Rothe (2016).

4.3. Pareto

The *Pareto criterion* says that the election method must respect unanimous opinions. There are two different versions of the Pareto criterion. The first version addresses situations with “ $a \succ_v b$ for all $v \in V$ ”, while the second version addresses situations with “ $a \succsim_v b$ for all $v \in V$ ” (for some pair of alternatives $a,b \in A$). The first version says that, when every voter strictly prefers alternative a to alternative b (i.e. $a \succ_v b$ for all $v \in V$), then alternative a must perform better than alternative b. The second version says that, when no voter strictly prefers alternative b to alternative a (i.e. $a \succsim_v b$ for all $v \in V$), then alternative b must not perform better than alternative a. We will prove that the Schulze method, as defined in section 2.2, satisfies both versions of the Pareto criterion.

4.3.1. Formulation #1

Definition:

An election method satisfies the first version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.1.1) $\forall\, v \in V$: $a \succ_v b$.

Then:

(4.3.1.2) $ab \in \mathcal{O}$.

(4.3.1.3) $\forall\, f \in A \setminus \{a,b\}$: $bf \in \mathcal{O} \Rightarrow af \in \mathcal{O}$.

(4.3.1.4) $\forall\, f \in A \setminus \{a,b\}$: $fa \in \mathcal{O} \Rightarrow fb \in \mathcal{O}$.

(4.3.1.5) $b \notin \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the Pareto criterion.

Proof:

With (2.1.1) and (4.3.1.1), we get

(4.3.1.6) $\forall\, e,f \in A$: $(N[a,b],N[b,a]) \succsim_D (N[e,f],N[f,e])$.

(4.3.1.7) $[\ (N[a,b],N[b,a]) \approx_D (N[e,f],N[f,e])\ ] \Leftrightarrow [\ \forall\, v \in V\text{: } e \succ_v f\ ]$.

With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab.

However, as we presumed that the individual ballots $\succ_v$ are strict weak orders, it is not possible that there is a directed cycle of unanimous opinions. Therefore, it is not possible that the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. Therefore, with (2.2.4), (4.3.1.6), and (4.3.1.7), we get (4.3.1.2). With (4.3.1.2), we get (4.3.1.5). With (4.3.1.2) and the transitivity of $\mathcal{O}$, we get (4.3.1.3) and (4.3.1.4). □

4.3.2. Formulation #2

Definition:

An election method satisfies the second version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.2.1) $\forall\, v \in V\colon a \succsim_v b.$

Then:

(4.3.2.2) $ba \notin \mathcal{O}.$

(4.3.2.3) $\forall\, f \in A \setminus \{a,b\}\colon bf \in \mathcal{O} \Rightarrow af \in \mathcal{O}.$

(4.3.2.4) $\forall\, f \in A \setminus \{a,b\}\colon fa \in \mathcal{O} \Rightarrow fb \in \mathcal{O}.$

(4.3.2.5) $b \in \mathcal{S} \Rightarrow a \in \mathcal{S}.$

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the Pareto criterion.

Proof:

With (4.3.2.1), we get

(4.3.2.6) $\forall\, e \in A \setminus \{a,b\}\colon N[a,e] \geq N[b,e].$

With (4.3.2.1), we get

(4.3.2.7) $\forall\, e \in A \setminus \{a,b\}\colon N[e,b] \geq N[e,a].$

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.8) $\forall\, e \in A \setminus \{a,b\}\colon (N[a,e],N[e,a]) \succsim_D (N[b,e],N[e,b]).$

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.9) $\forall\, e \in A \setminus \{a,b\}\colon (N[e,b],N[b,e]) \succsim_D (N[e,a],N[a,e]).$

Suppose $c(1),\dots,c(n) \in A$ is the strongest path from alternative b to alternative a. With (4.3.2.8) and (4.3.2.9), we get: $a,c(2),\dots,c(n-1),b$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.3.2.10) $P_D[a,b] \succsim_D P_D[b,a].$

With (4.3.2.10), we get (4.3.2.2).

Suppose $c(1),\dots,c(n) \in A$ is the strongest path from alternative b to alternative $f \in A \setminus \{a,b\}$. With (4.3.2.8), we get: $a,c(m+1),\dots,c(n)$, where $c(m)$

is the last occurrence of an alternative of the set $\{a,b\}$, is a path from alternative a to alternative f with at least the same strength. Therefore

(4.3.2.11) $\quad \forall\, f \in A \setminus \{a,b\}: P_D[a,f] \succsim_D P_D[b,f].$

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $f \in A \setminus \{a,b\}$ to alternative a. With (4.3.2.9), we get: $c(1),...,c(m-1),b$, where $c(m)$ is the first occurrence of an alternative of the set $\{a,b\}$, is a path from alternative f to alternative b with at least the same strength. Therefore

(4.3.2.12) $\quad \forall\, f \in A \setminus \{a,b\}: P_D[f,b] \succsim_D P_D[f,a].$

Part 1: Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13a) $\quad bf \in \mathcal{O}.$

With (4.3.2.13a), we get

(4.3.2.14a) $\quad P_D[b,f] \succ_D P_D[f,b].$

With (4.3.2.11), (4.3.2.14a), and (4.3.2.12), we get

(4.3.2.15a) $\quad P_D[a,f] \succsim_D P_D[b,f] \succ_D P_D[f,b] \succsim_D P_D[f,a].$

With (4.3.2.15a), we get (4.3.2.3).

Part 2: Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13b) $\quad fa \in \mathcal{O}.$

With (4.3.2.13b), we get

(4.3.2.14b) $\quad P_D[f,a] \succ_D P_D[a,f].$

With (4.3.2.12), (4.3.2.14b), and (4.3.2.11), we get

(4.3.2.15b) $\quad P_D[f,b] \succsim_D P_D[f,a] \succ_D P_D[a,f] \succsim_D P_D[b,f].$

With (4.3.2.15b), we get (4.3.2.4).

Part 3: Suppose

(4.3.2.13c) $\quad b \in \mathcal{S}.$

With (4.3.2.13c), we get

(4.3.2.14c) $\quad \forall\, f \in A \setminus \{b\}: fb \notin \mathcal{O}.$

With (4.3.2.4) and (4.3.2.14c), we get

(4.3.2.15c) $\quad \forall\, f \in A \setminus \{a,b\}: fa \notin \mathcal{O}.$

With (4.3.2.2) and (4.3.2.15c), we get

(4.3.2.16c) $\quad \forall\, f \in A \setminus \{a\}: fa \notin \mathcal{O}.$

With (4.3.2.16c), we get (4.3.2.5). □

4.4. Reversal Symmetry

Reversal symmetry as a criterion for single-winner election methods has been proposed by Saari (1994). This criterion says that, when $\succ_v$ is reversed for all $v \in V$, then also the result of the elections must be reversed; see (4.4.2). $\mathcal{S}^{\text{old}}$ must not be a strict subset of $\mathcal{S}^{\text{new}}$; $\mathcal{S}^{\text{new}}$ must not be a strict subset of $\mathcal{S}^{\text{old}}$; see (4.4.3). It should not be possible that the same alternatives are elected in the original situation and in the reversed situation, unless all alternatives are tied; see (4.4.4).

Basic idea of this criterion is that, when there is a vote on the best alternatives and then there is a vote on the worst alternatives and when in both cases the same alternatives are chosen, then this questions the logic of the underlying heuristic of the used election method.

<u>Definition:</u>

An election method satisfies *reversal symmetry* if the following holds:

Suppose:

(4.4.1) $\forall\, e,f \in A\ \forall\, v \in V\text{: } e \succ_v^{\text{old}} f \Leftrightarrow f \succ_v^{\text{new}} e.$

Then:

(4.4.2) $\forall\, a,b \in A\text{: } ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ba \in \mathcal{O}^{\text{new}}.$

(4.4.3) $(\ \exists\, i \in A\text{: } i \in \mathcal{S}^{\text{old}} \wedge i \notin \mathcal{S}^{\text{new}}\ ) \Leftrightarrow$
$(\ \exists\, j \in A\text{: } j \notin \mathcal{S}^{\text{old}} \wedge j \in \mathcal{S}^{\text{new}}\ ).$

(4.4.4) $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}} \Leftrightarrow \mathcal{S}^{\text{old}} = A.$

<u>Claim:</u>

The Schulze method, as defined in section 2.2, satisfies reversal symmetry.

Proof:

With (4.4.1), we get

(4.4.5) $\forall\, e,f \in A$: $N^{\text{old}}[e,f] = N^{\text{new}}[f,e]$.

With (4.4.5), we get

(4.4.6) $\forall\, e,f \in A$: $(N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \approx_D (N^{\text{new}}[f,e],N^{\text{new}}[e,f])$.

With (4.4.6), we get: When $c(1),...,c(n) \in A$ was a path from alternative $g \in A$ to alternative $h \in A \setminus \{g\}$, then $c(n),...,c(1)$ is a path from alternative h to alternative g with the same strength. Therefore

(4.4.7) $\forall\, g,h \in A$: $P_D^{\text{old}}[g,h] \approx_D P_D^{\text{new}}[h,g]$.

With (4.4.7), we get (4.4.2).

Part 1:

Suppose $\exists\, i \in A$: $i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$. With $i \notin \mathcal{S}^{\text{new}}$ and (4.1.14), we get that there is a $j \in \mathcal{S}^{\text{new}}$ with $ji \in \mathcal{O}^{\text{new}}$. With (4.4.2), we get $ij \in \mathcal{O}^{\text{old}}$ and, therefore, $j \notin \mathcal{S}^{\text{old}}$. With $j \notin \mathcal{S}^{\text{old}}$ and $j \in \mathcal{S}^{\text{new}}$, we get the “$\Rightarrow$” direction of (4.4.3). The proof for the “$\Leftarrow$” direction of (4.4.3) is analogous.

Part 2:

Suppose $\mathcal{S}^{\text{old}} = A$. Then we get $\mathcal{O}^{\text{old}} = \emptyset$. Otherwise, if there was an $ij \in \mathcal{O}^{\text{old}}$, we would immediately get $j \notin \mathcal{S}^{\text{old}}$ and, therefore, $\mathcal{S}^{\text{old}} \neq A$. With $\mathcal{O}^{\text{old}} = \emptyset$ and (4.4.2), we get $\mathcal{O}^{\text{new}} = \emptyset$ and, therefore, $\mathcal{S}^{\text{new}} = A$. With $\mathcal{S}^{\text{old}} = A$ and $\mathcal{S}^{\text{new}} = A$, we get $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}$.

Part 3:

Suppose $\mathcal{S}^{\text{old}} \neq A$. Then there is a $j \notin \mathcal{S}^{\text{old}}$. With (4.1.14), we get that there is an $i \in \mathcal{S}^{\text{old}}$ with $ij \in \mathcal{O}^{\text{old}}$. With (4.4.2), we get $ji \in \mathcal{O}^{\text{new}}$ and, therefore, $i \notin \mathcal{S}^{\text{new}}$. With $i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$, we get $\mathcal{S}^{\text{old}} \neq \mathcal{S}^{\text{new}}$. With part 2 and part 3, we get (4.4.4). □

4.5. Monotonicity

Monotonicity says that, when some voters rank alternative $a \in A$ higher [see (4.5.1) and (4.5.2)] without changing the order in which they rank the other alternatives relatively to each other [see (4.5.3)], then this must not hurt alternative a [see (4.5.4) – (4.5.6)]. Monotonicity is also known as *mono-raise* and *non-negative responsiveness*.

Definition:

An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified in such a manner that the following three statements are satisfied:

(4.5.1) $\forall\, f \in A \setminus \{a\}\ \forall\, v \in V$: $a \succ_v^{\text{old}} f \Rightarrow a \succ_v^{\text{new}} f$.

(4.5.2) $\forall\, f \in A \setminus \{a\}\ \forall\, v \in V$: $a \succsim_v^{\text{old}} f \Rightarrow a \succsim_v^{\text{new}} f$.

(4.5.3) $\forall\, e,f \in A \setminus \{a\}\ \forall\, v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f$.

Then:

(4.5.4) $\forall\, b \in A \setminus \{a\}$: $ab \in \mathcal{O}^{\text{old}} \Rightarrow ab \in \mathcal{O}^{\text{new}}$.

(4.5.5) $\forall\, b \in A \setminus \{a\}$: $ba \notin \mathcal{O}^{\text{old}} \Rightarrow ba \notin \mathcal{O}^{\text{new}}$.

(4.5.6) $a \in \mathcal{S}^{\text{old}} \Rightarrow a \in \mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies monotonicity.

Proof:

Part 1:

With (4.5.1), we get

(4.5.7) $\forall\, f \in A \setminus \{a\}$: $N^{\text{old}}[a,f] \leq N^{\text{new}}[a,f]$.

With (4.5.2), we get

(4.5.8) $\forall\, f \in A \setminus \{a\}$: $N^{\text{old}}[f,a] \geq N^{\text{new}}[f,a]$.

With (4.5.3), we get

(4.5.9) $\forall\, e,f \in A \setminus \{a\}$: $N^{\text{old}}[e,f] = N^{\text{new}}[e,f]$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.10) $\forall\, f \in A \setminus \{a\}$: $(N^{\text{old}}[a,f],N^{\text{old}}[f,a]) \precsim_D (N^{\text{new}}[a,f],N^{\text{new}}[f,a])$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.11) $\forall\ f \in A \setminus \{a\}$: $(N^{old}[f,a],N^{old}[a,f]) \gtrsim_D (N^{new}[f,a],N^{new}[a,f])$.

With (4.5.9), we get

(4.5.12) $\forall\ e,f \in A \setminus \{a\}$: $(N^{old}[e,f],N^{old}[f,e]) \approx_D (N^{new}[e,f],N^{new}[f,e])$.

Suppose $c(1),...,c(n) \in A$ was the strongest path from alternative a to alternative $b \in A \setminus \{a\}$. Then with (4.5.10) and (4.5.12), we get: $c(1),...,c(n)$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.5.13) $\forall\ b \in A \setminus \{a\}$: $P_D^{new}[a,b] \gtrsim_D P_D^{old}[a,b]$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $b \in A \setminus \{a\}$ to alternative a. Then with (4.5.11) and (4.5.12), we get: $c(1),...,c(n)$ was a path from alternative b to alternative a with at least the same strength. Therefore

(4.5.14) $\forall\ b \in A \setminus \{a\}$: $P_D^{old}[b,a] \gtrsim_D P_D^{new}[b,a]$.

With (4.5.13) and (4.5.14), we get (4.5.4) and (4.5.5).

Part 2:

It remains to prove (4.5.6). Suppose $a \in \mathcal{S}^{old}$. Then " $a \in \mathcal{S}^{new}$ " follows directly from (4.5.5). To prove " $\mathcal{S}^{new} \subseteq \mathcal{S}^{old}$ ", we have to prove: $h \notin \mathcal{S}^{old} \Rightarrow h \notin \mathcal{S}^{new}$.

As $a \in \mathcal{S}^{old}$, we get

(4.5.15) $\forall\ b \in A \setminus \{a\}$: $P_D^{old}[a,b] \gtrsim_D P_D^{old}[b,a]$.

Suppose $h \notin \mathcal{S}^{old}$. Then, according to (4.1.14), there must have been an alternative $g \in \mathcal{S}^{old}$ with

(4.5.16) $P_D^{old}[g,h] \succ_D P_D^{old}[h,g]$.

With (4.5.10) – (4.5.12) and (4.5.16), we get: $P_D^{new}[g,h] \succ_D P_D^{new}[h,g]$, unless at least one of the following two cases occurred.

Case 1: xa was a weakest link in the strongest path from alternative g to alternative h.

Case 2: ay was the weakest link in the strongest path from alternative h to alternative g.

With (4.5.15), we get: $P_D^{old}[a,h] \gtrsim_D P_D^{old}[h,a]$. For $P_D^{old}[a,h] \succ_D P_D^{old}[h,a]$, we would, with (4.5.4), immediately get $P_D^{new}[a,h] \succ_D P_D^{new}[h,a]$, so that alternative h is still not a potential winner. Therefore, without loss of generality, we can presume $g \in \mathcal{S}^{old} \setminus \{a\}$ and

(4.5.17) $P_D^{old}[a,h] \approx_D P_D^{old}[h,a]$.

With $a \in \mathcal{S}^{\text{old}}$ and $g \in \mathcal{S}^{\text{old}} \setminus \{a\}$, we get

(4.5.18) $P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$.

With (2.2.5), we get

(4.5.19) $\min_D \{ P_D^{\text{old}}[g,h], P_D^{\text{old}}[h,a] \} \precsim_D P_D^{\text{old}}[g,a]$.

(4.5.20) $\min_D \{ P_D^{\text{old}}[h,a], P_D^{\text{old}}[a,g] \} \precsim_D P_D^{\text{old}}[h,g]$.

Case 1: Suppose xa was a weakest link in the strongest path from alternative g to alternative h. Then

(4.5.21a) $P_D^{\text{old}}[g,h] \approx_D P_D^{\text{old}}[g,a]$ and

(4.5.22a) $P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h]$.

Now (4.5.18), (4.5.21a), and (4.5.16) give

(4.5.23a) $P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a] \approx_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]$,

while (4.5.17), (4.5.22a), and (4.5.16) give

(4.5.24a) $P_D^{\text{old}}[h,a] \approx_D P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]$.

But (4.5.23a) and (4.5.24a) together contradict (4.5.20).

Case 2: Suppose ay was the weakest link in the strongest path from alternative h to alternative g. Then

(4.5.21b) $P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g]$ and

(4.5.22b) $P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g]$.

Now (4.5.22b), (4.5.21b), and (4.5.18) give

(4.5.23b) $P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$,

while (4.5.16), (4.5.21b), and (4.5.18) give

(4.5.24b) $P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \approx_D P_D^{\text{old}}[g,a]$.

But (4.5.23b) and (4.5.24b) together contradict (4.5.19).

We have proven that neither case 1 nor case 2 is possible. Therefore

(4.5.25) $P_D^{\text{new}}[g,h] \succ_D P_D^{\text{new}}[h,g]$.

With (4.5.25), we get: $h \notin \mathcal{S}^{\text{new}}$. □

4.6. Independence of Clones

Independence of clones as a criterion for single-winner election methods has been proposed by Tideman (1987). This criterion says that running a large number of similar alternatives, so-called *clones*, must not have any impact on the result of the elections.

The precise definition for a *set of clones* stipulates that every voters ranks all the alternatives of this set in a consecutive manner; see (4.6.1) and (4.6.2). Replacing an alternative $d \in A^{\text{old}}$ by a set of clones K should not change the winner; see (4.6.7) and (4.6.8).

This criterion is very desirable especially for referendums because, while it might be difficult to find several candidates who are simultaneously sufficiently popular to campaign with them and sufficiently similar to misuse them for this strategy, it is usually very simple to formulate a large number of almost identical proposals. For example: In 1969, when the Canadian city that is now known as *Thunder Bay* was amalgamating, there was some controversy over what the name should be. In opinion polls, a majority of the voters preferred the name *The Lakehead* to the name *Thunder Bay*. But when the polls opened, there were three names on the referendum ballot: *Thunder Bay*, *Lakehead*, and *The Lakehead*. As the ballots were counted using *plurality voting*, it was not a surprise when *Thunder Bay* won. The votes were as follows: *Thunder Bay* 15870, *Lakehead* 15302, *The Lakehead* 8377 (Cretney, 2000).

Definition:

An election method is *independent of clones* if the following holds:

Suppose $d \in A^{\text{old}}$. Suppose $A^{\text{new}} := (A^{\text{old}} \cup K) \setminus \{d\}$.

Suppose alternative d is replaced by the set of alternatives K in such a manner that the following three statements are satisfied:

(4.6.1) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V\colon e \succ_v^{\text{old}} d \Leftrightarrow e \succ_v^{\text{new}} g.$

(4.6.2) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V\colon d \succ_v^{\text{old}} f \Leftrightarrow g \succ_v^{\text{new}} f.$

(4.6.3) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}\ \forall\, v \in V\colon e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f.$

Then the following statements are satisfied:

(4.6.4) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\colon ad \in \mathcal{O}^{\text{old}} \Leftrightarrow ag \in \mathcal{O}^{\text{new}}.$

(4.6.5) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\colon db \in \mathcal{O}^{\text{old}} \Leftrightarrow gb \in \mathcal{O}^{\text{new}}.$

(4.6.6) $\forall\, a,b \in A^{\text{old}} \setminus \{d\}\colon ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ab \in \mathcal{O}^{\text{new}}.$

(4.6.7) $d \in \mathcal{S}^{\text{old}} \Leftrightarrow \mathcal{S}^{\text{new}} \cap K \neq \emptyset.$

(4.6.8) $\forall\, a \in A^{\text{old}} \setminus \{d\}\colon a \in \mathcal{S}^{\text{old}} \Leftrightarrow a \in \mathcal{S}^{\text{new}}.$

Claim:

The Schulze method, as defined in section 2.2, is independent of clones.

Proof:

With (4.6.1), we get

(4.6.9) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $N^{\text{old}}[e,d] = N^{\text{new}}[e,g]$.

With (4.6.2), we get

(4.6.10) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $N^{\text{old}}[d,f] = N^{\text{new}}[g,f]$.

With (4.6.3), we get

(4.6.11) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}$: $N^{\text{old}}[e,f] = N^{\text{new}}[e,f]$.

With (4.6.9) and (4.6.10), we get

(4.6.12) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $(N^{\text{old}}[e,d],N^{\text{old}}[d,e]) \approx_D (N^{\text{new}}[e,g],N^{\text{new}}[g,e])$.

With (4.6.9) and (4.6.10), we get

(4.6.13) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $(N^{\text{old}}[d,f],N^{\text{old}}[f,d]) \approx_D (N^{\text{new}}[g,f],N^{\text{new}}[f,g])$.

With (4.6.11), we get

(4.6.14) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}$: $(N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \approx_D (N^{\text{new}}[e,f],N^{\text{new}}[f,e])$.

Suppose $c(1),...,c(n) \in A^{\text{old}}$ was the strongest path from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative d. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(n-1),g$ is a path from alternative a to alternative $g \in K$ with the same strength. Therefore

(4.6.15) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{new}}[a,g] \succsim_D P_D^{\text{old}}[a,d]$.

Suppose $c(1),...,c(n) \in A^{\text{new}}$ is the strongest path from alternative $a \in A^{\text{new}} \setminus K$ to alternative $g \in K$. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(m-1),d$, where $c(m)$ is the first occurrence of an alternative of the set K, was a path from alternative a to alternative d with at least the same strength. Therefore

(4.6.16) $\forall\, a \in A^{\text{new}} \setminus K\ \forall\, g \in K$: $P_D^{\text{old}}[a,d] \succsim_D P_D^{\text{new}}[a,g]$.

Suppose $c(1),...,c(n) \in A^{\text{old}}$ was the strongest path from alternative d to alternative $b \in A^{\text{old}} \setminus \{d\}$. Then with (4.6.13) and (4.6.14), we get: $g,c(2),...,c(n)$ is a path from alternative $g \in K$ to alternative b with the same strength. Therefore

(4.6.17) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{new}}[g,b] \succsim_D P_D^{\text{old}}[d,b]$.

Suppose $c(1),...,c(n) \in A^{\text{new}}$ is the strongest path from alternative $g \in K$ to alternative $b \in A^{\text{new}} \setminus K$. Then with (4.6.13) and (4.6.14), we get: $d,c(m+1),...,c(n)$, where $c(m)$ is the last occurrence of an alternative of the set K, was a path from alternative d to alternative b with at least the same strength. Therefore

(4.6.18) $\forall\, b \in A^{\text{new}} \setminus K\ \forall\, g \in K$: $P_D^{\text{old}}[d,b] \succsim_D P_D^{\text{new}}[g,b]$.

(α) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ did not contain alternative d. Then with (4.6.14), we get: $c(1),...,c(n)$ is still a path from alternative a to alternative b with the same strength. Therefore: $P^{\text{new}}_{D}[a,b] \gtrsim_D P^{\text{old}}_{D}[a,b]$.

(β) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ contained alternative d. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(n)$, with alternative d replaced by an arbitrarily chosen alternative $g \in K$, is still a path from alternative a to alternative b with the same strength. Therefore: $P^{\text{new}}_{D}[a,b] \gtrsim_D P^{\text{old}}_{D}[a,b]$.

With (α) and (β), we get

(4.6.19) $\quad \forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P^{\text{new}}_{D}[a,b] \gtrsim_D P^{\text{old}}_{D}[a,b]$.

(γ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ does not contain alternatives of the set K. Then with (4.6.14), we get: $c(1),...,c(n)$ was a path from alternative a to alternative b with the same strength. Therefore: $P^{\text{old}}_{D}[a,b] \gtrsim_D P^{\text{new}}_{D}[a,b]$.

(δ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ contains some alternatives of the set K. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(s-1),d,c(t+1),...,c(n)$, where $c(s)$ is the first occurrence of an alternative of the set K and $c(t)$ is the last occurrence of an alternative of the set K, was a path from alternative a to alternative b with at least the same strength. Therefore: $P^{\text{old}}_{D}[a,b] \gtrsim_D P^{\text{new}}_{D}[a,b]$.

With (γ) and (δ), we get

(4.6.20) $\quad \forall\, a,b \in A^{\text{new}} \setminus K$: $P^{\text{old}}_{D}[a,b] \gtrsim_D P^{\text{new}}_{D}[a,b]$.

Combining (4.6.15) and (4.6.16) gives

(4.6.21) $\quad \forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P^{\text{old}}_{D}[a,d] \approx_D P^{\text{new}}_{D}[a,g]$.

Combining (4.6.17) and (4.6.18) gives

(4.6.22) $\quad \forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P^{\text{old}}_{D}[d,b] \approx_D P^{\text{new}}_{D}[g,b]$.

Combining (4.6.19) and (4.6.20) gives

(4.6.23) $\quad \forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P^{\text{old}}_{D}[a,b] \approx_D P^{\text{new}}_{D}[a,b]$.

With (4.6.21) – (4.6.23), we get (4.6.4) – (4.6.6).

Part 1:

Suppose $d \in \mathcal{S}^{\text{old}}$. Then

(4.6.24) $\quad \forall\, a \in A^{\text{old}} \setminus \{d\}$: $ad \notin \mathcal{O}^{\text{old}}$.

With (4.6.4) and (4.6.24), we get

(4.6.25) $\forall\, a \in A^{\text{new}} \setminus K\ \forall\, g \in K\text{: } ag \notin \mathcal{O}^{\text{new}}$.

Since the binary relation $\mathcal{O}^{\text{new}}$, as defined in (2.2.1), is asymmetric and transitive, there must be an alternative $k \in K$ with

(4.6.26) $\forall\, l \in K \setminus \{k\}\text{: } lk \notin \mathcal{O}^{\text{new}}$.

With (4.6.25) and (4.6.26), we get $k \in \mathcal{S}^{\text{new}} \cap K$ and, therefore, $\mathcal{S}^{\text{new}} \cap K \neq \emptyset$.

Part 2:

Suppose $d \notin \mathcal{S}^{\text{old}}$. Then

(4.6.27) $\exists\, a \in A^{\text{old}} \setminus \{d\}\text{: } ad \in \mathcal{O}^{\text{old}}$.

With (4.6.4) and (4.6.27), we get

(4.6.28) $\exists\, a \in A^{\text{new}} \setminus K\ \forall\, g \in K\text{: } ag \in \mathcal{O}^{\text{new}}$.

With (4.6.28), we get: $\mathcal{S}^{\text{new}} \cap K = \emptyset$.

With part 1 and part 2, we get (4.6.7).

Part 3:

Suppose $a \in A^{\text{old}} \setminus \{d\}$ and $a \in \mathcal{S}^{\text{old}}$. Then

(4.6.29) $da \notin \mathcal{O}^{\text{old}}$.

(4.6.30) $\forall\, b \in A^{\text{old}} \setminus \{a,d\}\text{: } ba \notin \mathcal{O}^{\text{old}}$.

With (4.6.5) and (4.6.29), we get

(4.6.31) $\forall\, g \in K\text{: } ga \notin \mathcal{O}^{\text{new}}$.

With (4.6.6) and (4.6.30), we get

(4.6.32) $\forall\, b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)\text{: } ba \notin \mathcal{O}^{\text{new}}$.

With (4.6.31) and (4.6.32), we get: $a \in \mathcal{S}^{\text{new}}$.

Part 4:

Suppose $a \in A^{\text{old}} \setminus \{d\}$ and $a \notin \mathcal{S}^{\text{old}}$. Then at least one of the following two statements must have been valid:

(4.6.33a) $da \in \mathcal{O}^{\text{old}}$.

(4.6.33b) $\exists\, b \in A^{\text{old}} \setminus \{a,d\}\text{: } ba \in \mathcal{O}^{\text{old}}$.

With (4.6.5), (4.6.6), and (4.6.33), we get that at least one of the following two statements must be valid:

(4.6.34a) $\forall\, g \in K\text{: } ga \in \mathcal{O}^{\text{new}}$.

(4.6.34b) $\exists\, b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)\text{: } ba \in \mathcal{O}^{\text{new}}$.

With (4.6.34), we get: $a \notin \mathcal{S}^{\text{new}}$.

With part 3 and part 4, we get (4.6.8). □

4.7. Smith Criterion, Condorcet Winners, Condorcet Losers

The *Smith criterion* and *Smith-IIA* (where IIA means “independence of irrelevant alternatives”) say that *weak* alternatives should have no impact on the result of the elections.

Suppose:

(4.7.1) $\emptyset \neq B_1 \subsetneq A$, $\emptyset \neq B_2 \subsetneq A$, $B_1 \cup B_2 = A$, $B_1 \cap B_2 = \emptyset$.

(4.7.2) $\forall\ a \in B_1\ \forall\ b \in B_2$: $N[a,b] > N[b,a]$.

Then a *weak* alternative in the Smith paradigm is an alternative $b \in B_2$. Adding or removing a weak alternative $b \in B_2$ should have no impact on the set $\mathcal{S}$ of potential winners.

Definition:

An election method satisfies the *Smith criterion* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.3) $\forall\ a \in B_1\ \forall\ b \in B_2$: $ab \in \mathcal{O}$.

(4.7.4) $\emptyset \neq \mathcal{S} \subseteq B_1$.

Remark:

If B_1 consists of only one alternative $a \in A$, then this alternative is the so-called *Condorcet winner* and the Smith criterion becomes the so-called *Condorcet criterion* (Condorcet, 1785). In short:

(4.7.5) Alternative $a \in A$ is a *Condorcet winner* $:\Leftrightarrow$
$N[a,b] > N[b,a]$ for all $b \in A \setminus \{a\}$.

(4.7.6) An election method satisfies the *Condorcet criterion* if the following holds:

Alternative $a \in A$ is a Condorcet winner. $\Rightarrow \mathcal{S} = \{a\}$.

If B_2 consists of only one alternative $b \in A$, then this alternative is the so-called *Condorcet loser* and the Smith criterion becomes the so-called *Condorcet loser criterion*. In short:

(4.7.7) Alternative $b \in A$ is a *Condorcet loser* $:\Leftrightarrow$
$N[a,b] > N[b,a]$ for all $a \in A \setminus \{b\}$.

(4.7.8) An election method satisfies the *Condorcet loser criterion* if the following holds:

Alternative $b \in A$ is a Condorcet loser. $\Rightarrow b \notin \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the Smith criterion.

Proof:

The proof is trivial. Presumption (2.1.5) guarantees that any pairwise victory is stronger than any pairwise defeat. If $a \in B_1$ and $b \in B_2$, then already the link ab is a path from alternative a to alternative b that consists only of a pairwise victory. On the other side, (4.7.2) says that there cannot be a path from alternative b to alternative a that contains no pairwise defeat. So already the link ab is stronger than any path from alternative b to alternative a. □

Definition:

An election method satisfies *Smith-IIA* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.9) If $d \in B_2$ is removed, then

(a) $\forall\ e,f \in B_1$: $ef \in \mathcal{O}^{\text{old}} \Leftrightarrow ef \in \mathcal{O}^{\text{new}}$.

(b) $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}$.

(4.7.10) If $d \in B_1$ is removed, then

$\forall\ e,f \in B_2$: $ef \in \mathcal{O}^{\text{old}} \Leftrightarrow ef \in \mathcal{O}^{\text{new}}$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies Smith-IIA.

Proof:

We will prove (4.7.9)(a). The proof for (4.7.10) is analogous.

(4.7.9)(b) follows directly from (4.7.4) and (4.7.9)(a).

Part 1: Suppose $e,f \in B_1$. Suppose $ef \in \mathcal{O}^{\text{old}}$. Then

(4.7.11) $P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e]$.

With (2.2.3), we get

(4.7.12) $P_D^{\text{old}}[e,f] \gtrsim_D (N[e,f],N[f,e])$.

With (4.7.11) and (2.2.3), we get

(4.7.13) $P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e] \gtrsim_D (N[f,e],N[e,f])$.

With (4.7.12) and (4.7.13), we get

(4.7.14) $P_D^{\text{old}}[e,f] \gtrsim_D \max_D \{ (N[e,f],N[f,e]), (N[f,e],N[e,f]) \}$.

With (4.7.2), we get: Any path from alternative $e \in B_1$ to alternative $f \in B_1$ that contained alternative $d \in B_2$ necessarily contained a pairwise defeat.

As it is not possible that the link ef is a pairwise defeat and that simultaneously the link fe is a pairwise defeat, $\max_D \{ (N[e,f],N[f,e]), (N[f,e], N[e,f]) \}$ is stronger than any pairwise defeat [because of (2.1.5)]. Therefore, with (4.7.2) and (4.7.14), we get: The strongest path from alternative $e \in B_1$ to alternative $f \in B_1$ did not contain alternative $d \in B_2$. Therefore

(4.7.15) $P_D^{\text{new}}[e,f] \approx_D P_D^{\text{old}}[e,f]$.

As the elimination of alternative $d \in B_2$ only removes paths, we get

(4.7.16) $P_D^{\text{new}}[f,e] \lesssim_D P_D^{\text{old}}[f,e]$.

With (4.7.15), (4.7.11), and (4.7.16), we get

(4.7.17) $P_D^{\text{new}}[e,f] \approx_D P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e] \gtrsim_D P_D^{\text{new}}[f,e]$.

With (4.7.17), we get: $ef \in \mathcal{O}^{\text{new}}$.

Part 2: The proof for “ $P_D^{\text{old}}[f,e] \succ_D P_D^{\text{old}}[e,f]$ ” is analogous.

Part 3: When we have $P^{\text{old}}_{D}[e,f] \approx_D P^{\text{old}}_{D}[f,e]$ then, with the same argumentation as in Part 1, we get

(4.7.18) $P^{\text{old}}_{D}[e,f] \gtrsim_D \max_D \{ (N[e,f],N[f,e]), (N[f,e],N[e,f]) \}$.

(4.7.19) $P^{\text{old}}_{D}[f,e] \gtrsim_D \max_D \{ (N[e,f],N[f,e]), (N[f,e],N[e,f]) \}$.

So with the same argumentation as in Part 1, we can show that neither the strongest path from alternative $e \in B_1$ to alternative $f \in B_1$ nor the strongest path from alternative $f \in B_1$ to alternative $e \in B_1$ did contain alternative $d \in B_2$. □

The *majority criterion for solid coalitions* says that, when a majority of the voters strictly prefers every alternative of a given set of alternatives to every alternative outside this set of alternatives, then the winner must be chosen from this set. In short, an election method satisfies the *majority criterion for solid coalitions* if the following holds:

Suppose (4.7.1).
Suppose $\| \{ v \in V \mid \forall a \in B_1 \ \forall b \in B_2 \colon a \succ_v b \} \| > N/2$.
Then $\mathcal{S} \subseteq B_1$.

If B_1 consists of only one alternative $a \in A$, then this is the so-called *majority criterion*. If B_2 consists of only one alternative $b \in A$, then this is the so-called *majority loser criterion*.

Participation says that adding a list W of ballots, on which every alternative of a given set of alternatives is strictly preferred to every alternative outside this set, must not hurt the alternatives of this set. In short, an election method satisfies *participation* if the following holds:

Suppose (4.7.1).

Suppose $\forall a \in B_1 \ \forall b \in B_2 \ \forall w \in W \colon a \succ_w b$.

Suppose $V^{\text{new}} := V^{\text{old}} + W$.

Then (4.7.20) $\forall e \in B_1 \ \forall f \in B_2 \colon ef \in \mathcal{O}^{\text{old}} \Rightarrow ef \in \mathcal{O}^{\text{new}}$.

(4.7.21) $\forall e \in B_1 \ \forall f \in B_2 \colon fe \notin \mathcal{O}^{\text{old}} \Rightarrow fe \notin \mathcal{O}^{\text{new}}$.

(4.7.22) $\mathcal{S}^{\text{old}} \cap B_1 \neq \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_1 \neq \emptyset$.

(4.7.23) $\mathcal{S}^{\text{old}} \cap B_2 = \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_2 = \emptyset$.

The Smith criterion implies the majority criterion for solid coalitions, the Condorcet criterion, and the Condorcet loser criterion. The majority criterion for solid coalitions implies the majority criterion and the majority loser criterion. The Condorcet criterion implies the majority criterion. The Condorcet loser criterion implies the majority loser criterion. Unfortunately, the Condorcet criterion is incompatible with the participation criterion (Moulin, 1988). Example 5 shows a drastic violation of the participation criterion.

4.8. MinMax Set

For all $\emptyset \neq B \subsetneq A$, we define

(4.8.1) $\Gamma_D(B) := \max_D \{ (N[x,y],N[y,x]) \mid x \notin B, y \in B \}$.

Furthermore, we define

(4.8.2) $\beta_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \}$.

(4.8.3) $\mathfrak{B}_D := \cup \{ \emptyset \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$.

$\mathfrak{B}_D$ is the *MinMax set*. $\mathfrak{B}_D$ has the following properties:

1. $\mathfrak{B}_D \neq \emptyset$.

2. If $\mathfrak{B}_D$ consists of only one alternative $a \in A$, then alternative a is the unique Simpson-Kramer winner (i.e. that alternative $a \in A$ with minimum $\max_D \{ (N[b,a],N[a,b]) \mid b \in A \setminus \{a\} \}$).

3. If $d \in \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{\text{new}} = (\mathfrak{B}_D \cup K) \setminus \{d\}$.

4. If $d \notin \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{\text{new}} = \mathfrak{B}_D$.

So, in some sense, the MinMax set $\mathfrak{B}_D$ is a clone-proof generalization of the Simpson-Kramer winner.

When we want primarily that the used election method is independent of clones and secondarily that the strongest link *ef*, that is overruled when determining the winner, is minimized, then we have to demand that the winner is always chosen from the MinMax set $\mathfrak{B}_D$.

<u>Claim:</u>

The Schulze method, as defined in section 2.2, has the following properties:

(4.8.4) $\forall\, a \in \mathfrak{B}_D\ \forall\, b \notin \mathfrak{B}_D$: $ab \in \mathcal{O}$.

(4.8.5) $\mathcal{S} \subseteq \mathfrak{B}_D$.

<u>Proof:</u>

Suppose $a \in \mathfrak{B}_D$. Then we get

(4.8.6) $\exists\, \emptyset \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$.

Suppose $b \notin \mathfrak{B}_D$. Then we get

(4.8.7) $\gamma_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \text{ and } b \in B \} \succ_D \beta_D$.

We will prove the following claims:

Claim #1: $P_D[b,a] \precsim_D \beta_D$.
Claim #2: $P_D[a,b] \succsim_D \gamma_D$.

With claim #1, claim #2, and (4.8.7), we get

(4.8.8) $P_D[a,b] \succsim_D \gamma_D \succ_D \beta_D \succsim_D P_D[b,a]$.

With (4.8.8), we get (4.8.4). With (4.8.4), we get (4.8.5).

Proof of claim #1:

With (4.8.6) and (4.8.7), we get

(4.8.9) $\exists\ \varnothing \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$ and $b \notin B$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative b to alternative a. Suppose $c(i)$ is the last alternative with $c(i) \notin B$. Then we get $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \precsim_D \beta_D$. Therefore, we get

(4.8.10) $P_D[b,a] \precsim_D \beta_D$.

Proof of claim #2:

We can construct a path from alternative a to alternative b with a strength of at least γ_D as follows:

(1) We start with $E_1 := \{a\}$ and $i := 1$. Trivially, we get $b \notin E_1$ and $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_1 \setminus \{a\}$.

(2) At each stage, we consider the set $B_i := A \setminus E_i$.

With $b \in B_i$ and with (4.8.7), we get

(4.8.11) $\Gamma_D(B_i) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We choose $f \in E_i$ and $g \in B_i$ with

(4.8.12) $(N[f,g],N[g,f]) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We define $E_{i+1} := E_i \cup \{g\}$.

With $f \in E_i$, with $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_i \setminus \{a\}$, with $(N[f,g], N[g,f]) \succsim_D \gamma_D$, and with $E_{i+1} := E_i \cup \{g\}$, we get

(4.8.13) $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_{i+1} \setminus \{a\}$.

(3) We repeat stage 2 with $i \rightarrow i+1$, until $g \equiv b$.

Therefore, we get

(4.8.14) $P_D[a,b] \succsim_D \gamma_D$. □

Example 6 shows that IPDA and the desideratum, that the winner is always chosen from the MinMax set $\mathcal{B}_D$, are incompatible. In example 6(old), we get $\mathcal{B}_D^{\text{old}} = \{a, c, d\}$. In example 6(new), we get $\mathcal{B}_D^{\text{new}} = \{b\}$. Therefore, $\mathcal{B}_D^{\text{old}} \cap \mathcal{B}_D^{\text{new}} = \emptyset$. Thus, the desideratum, that the winner is always chosen from the MinMax set $\mathcal{B}_D$, implies that the winner is changed.

Actually, the Schulze method can be described completely with the desideratum to find a binary relation $\mathcal{O}$ on A that, primarily, is independent of clones (as defined in section 4.6) and that, secondarily, tries to rank the alternatives according to their worst defeats.

For all $a,b \in A$, we define

(4.8.15) $\gamma_D[a,b] : = \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \text{ and } a \notin B \text{ and } b \in B \}$.

(4.8.16) $ab \in \mathcal{O} :\Leftrightarrow \gamma_D[a,b] \succ_D \gamma_D[b,a]$.

To prove that (4.8.16) is identical to (2.2.1), we have to prove $\gamma_D[a,b] = P_D[a,b]$. This proof is identical to the proof for (4.8.4).

Example 1

In example 1 (section 3.1), we have:

$\Gamma_D(B) : = \max_D \{ (N[x,y],N[y,x]) \mid x \notin B, y \in B \}$.

$\Gamma_D(\{a\}) = (13,8)$.
$\Gamma_D(\{b\}) = (19,2)$.
$\Gamma_D(\{c\}) = (14,7)$.
$\Gamma_D(\{d\}) = (12,9)$.
$\Gamma_D(\{a,b\}) = (19,2)$.
$\Gamma_D(\{a,c\}) = (13,8)$.
$\Gamma_D(\{a,d\}) = (13,8)$.
$\Gamma_D(\{b,c\}) = (19,2)$.
$\Gamma_D(\{b,d\}) = (15,6)$.
$\Gamma_D(\{c,d\}) = (14,7)$.
$\Gamma_D(\{a,b,c\}) = (19,2)$.
$\Gamma_D(\{a,b,d\}) = (15,6)$.
$\Gamma_D(\{a,c,d\}) = (13,8)$.
$\Gamma_D(\{b,c,d\}) = (14,7)$.

$\beta_D : = \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \}$.

$\beta_D = (12,9)$.

$\mathcal{B}_D : = \cup \{ \emptyset \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$.

$\mathcal{B}_D = \{d\}$.

So with (4.8.5), we get $\mathcal{S} = \{d\}$.

$\gamma_D[x,y] := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \text{ and } x \notin B \text{ and } y \in B \}$.

$\gamma_D[a,b] = \Gamma_D(\{b,c,d\}) = (14,7)$.
$\gamma_D[a,c] = \Gamma_D(\{c\}) = \Gamma_D(\{c,d\}) = \Gamma_D(\{b,c,d\}) = (14,7)$.
$\gamma_D[a,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[b,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,c\}) = \Gamma_D(\{a,d\}) = \Gamma_D(\{a,c,d\}) = (13,8)$.
$\gamma_D[b,c] = \Gamma_D(\{a,c\}) = \Gamma_D(\{a,c,d\}) = (13,8)$.
$\gamma_D[b,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[c,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,d\}) = (13,8)$.
$\gamma_D[c,b] = \Gamma_D(\{b,d\}) = \Gamma_D(\{a,b,d\}) = (15,6)$.
$\gamma_D[c,d] = \Gamma_D(\{d\}) = (12,9)$.
$\gamma_D[d,a] = \Gamma_D(\{a\}) = \Gamma_D(\{a,c\}) = (13,8)$.
$\gamma_D[d,b] = \Gamma_D(\{b\}) = \Gamma_D(\{a,b\}) = \Gamma_D(\{b,c\}) = \Gamma_D(\{a,b,c\}) = (19,2)$.
$\gamma_D[d,c] = \Gamma_D(\{a,c\}) = (13,8)$.

4.9. Prudence

Prudence as a criterion for single-winner election methods has been proposed by Köhler (1978) and generalized by Arrow and Raynaud (1986). This criterion says that the strength λ_D of the strongest link *ef*, that is not respected by the binary relation $\mathcal{O}$, should be as weak as possible. So $\lambda_D := \max_D \{ (N[e,f],N[f,e]) \mid ef \notin \mathcal{O} \}$ should be minimized.

A *directed cycle* is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:

1. $c(1) \equiv c(n)$.
2. $n \in \mathbb{N}$ with $3 \leq n < \infty$.
3. For all $i = 1,...,(n-1)$: $c(i+1) \in A \setminus \{c(i)\}$.

It is obvious that, when there is a directed cycle $c(1),...,c(n)$, then the strongest link, that is not respected by the binary relation $\mathcal{O}$, is at least as strong as the weakest link $c(i),c(i+1)$ of this directed cycle. Therefore, we get:

(4.9.1) $\qquad \lambda_D \succsim_D \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \}$.

As we have to make this consideration for all directed cycles, the maximum, that we can ask for, is the following criterion.

<u>Definition:</u>

Suppose $\lambda_D \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest directed cycle.

(4.9.2) $$\lambda_D := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \} \mid c(1),...,c(n) \text{ is a directed cycle } \}.$$

Then an election method is *prudent* if the following holds:

(4.9.3) $\qquad \forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow ab \in \mathcal{O}$.

(4.9.4) $\qquad \forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow b \notin \mathcal{S}$.

Claim:

The Schulze method, as defined in section 2.2, is prudent.

Proof:

The proof is trivial. With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. □

Example 1

In example 1 (section 3.1), the strongest directed cycle (measured by the strength of its weakest link) is a,(14,7),c,(15,6),b,(13,8),a with a strength of $\lambda_D \approx_D (13,8)$. So prudence says that the collective ranking $\mathcal{O}$ must respect all links that are stronger than (13,8).

$(N[d,b],N[b,d]) = (19,2) \succ_D (13,8) \approx_D \lambda_D \Rightarrow db \in \mathcal{O}$.

$(N[c,b],N[b,c]) = (15,6) \succ_D (13,8) \approx_D \lambda_D \Rightarrow cb \in \mathcal{O}$.

$(N[a,c],N[c,a]) = (14,7) \succ_D (13,8) \approx_D \lambda_D \Rightarrow ac \in \mathcal{O}$.

With $db \in \mathcal{O}$, $cb \in \mathcal{O}$, and $ac \in \mathcal{O}$, we get $b \notin \mathcal{S}$ and $c \notin \mathcal{S}$.

4.10. Schwartz

The Schwartz criterion as a criterion for single-winner election methods has been proposed by Schwartz (1986). The Schwartz criterion implies the Smith criterion.

A *chain* from alternative $x \in A$ to alternative $y \in A$ is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:

1. $x \equiv c(1)$.
2. $y \equiv c(n)$.
3. $2 \leq n < \infty$.
4. For all $i = 1,...,(n-1)$: $c(i+1) \in A \setminus \{c(i)\}$.
5. For all $i = 1,...,(n-1)$: $N[c(i),c(i+1)] > N[c(i+1),c(i)]$.

Definition:

An election method satisfies the *Schwartz criterion* if the following holds:

> Suppose there is a chain from alternative $a \in A$ to alternative $b \in A$ and no chain from alternative b to alternative a. Then:
>
> (4.10.1) $ab \in \mathcal{O}$.
>
> (4.10.2) $b \notin \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the Schwartz criterion.

Proof:

The proof is trivial. □

4.11. Weak Condorcet Winners and Weak Condorcet Losers

4.11.1. Weak Condorcet Winners

A *Condorcet winner* is an alternative $a \in A$ that wins every head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.1.1) Alternative $a \in A$ is a *Condorcet winner* : $\Leftrightarrow$
$N[a,b] > N[b,a]$ for all $b \in A \setminus \{a\}$.

A *weak Condorcet winner* is an alternative $a \in A$ that doesn't lose any head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.1.2) Alternative $a \in A$ is a *weak Condorcet winner* : $\Leftrightarrow$
$N[a,b] \geq N[b,a]$ for all $b \in A \setminus \{a\}$.

Suppose $\mathcal{E}$ is the set of weak Condorcet winners. Then we get:

(4.11.1.3) $a \in \mathcal{E} :\Leftrightarrow N[a,b] \geq N[b,a]$ for all $b \in A \setminus \{a\}$.

A frequently stated desideratum says that, when there is a weak Condorcet winner, then it should win.

When there happens to be exactly one potential winner $x \in A$ and exactly one weak Condorcet winner $y \in A$, it is obvious what the above desideratum means: Alternative x and alternative y must be the same alternative.

In other words:

(4.11.1.4) $\quad | \mathcal{E} | = 1$ and $| \mathcal{S} | = 1 \Rightarrow \mathcal{E} = \mathcal{S}$.

However, when there happens to be more than one potential winner or more than one weak Condorcet winner, the proper formulation for the above desideratum isn’t obvious. The most intuitive formulation is:

(4.11.1.5) $\quad \mathcal{E} \neq \emptyset \Rightarrow \mathcal{S} \subseteq \mathcal{E}$.

Formulation (4.11.1.5) says that, when there is at least one weak Condorcet winner, then every potential winner should be a weak Condorcet winner. Unfortunately, the following example demonstrates that (4.11.1.5) is incompatible with reversal symmetry:

> Suppose there are four alternatives $A = \{a,b,c,d\}$. Suppose $N^{old}[a,b] = N^{old}[b,a]$, $N^{old}[a,c] = N^{old}[c,a]$, $N^{old}[a,d] = N^{old}[d,a]$, $N^{old}[b,c] > N^{old}[c,b]$, $N^{old}[c,d] > N^{old}[d,c]$, and $N^{old}[d,b] > N^{old}[b,d]$. Then we get $\mathcal{E}^{old} = \{a\}$. With (4.11.1.5) and the requirement that $\mathcal{S}^{old}$ must not be empty, we get $\mathcal{S}^{old} = \{a\}$.
>
> When the individual preferences are reversed, as defined in (4.4.1), we get $N^{new}[a,b] = N^{new}[b,a]$, $N^{new}[a,c] = N^{new}[c,a]$, $N^{new}[a,d] = N^{new}[d,a]$, $N^{new}[b,c] < N^{new}[c,b]$, $N^{new}[c,d] < N^{new}[d,c]$, and $N^{new}[d,b] < N^{new}[b,d]$. Therefore, we get $\mathcal{E}^{new} = \{a\}$. With (4.11.1.5) and the requirement that $\mathcal{S}^{new}$ must not be empty, we get $\mathcal{S}^{new} = \{a\}$.
>
> But $\mathcal{S}^{old} = \{a\}$ and $\mathcal{S}^{new} = \{a\}$ together contradict (4.4.4).

In short: It can happen that the same alternative is the unique weak Condorcet winner in the original situation and, simultaneously, the unique weak Condorcet winner in the reversed situation. Therefore, (4.11.1.5) cannot be compatible with reversal symmetry.

Furthermore, the following example demonstrates that (4.11.1.5) is incompatible with independence of clones:

> Suppose there are only two alternatives $A^{old} = \{a,b\}$. Suppose $N[a,b] = N[b,a]$. Then we get $\mathcal{E}^{old} = \{a,b\}$. With (4.11.1.5), we get $\mathcal{S}^{old} \subseteq \{a,b\}$.
>
> Case I: Suppose $a \in \mathcal{S}^{old}$. When alternative a is replaced by alternatives a_1,a_2,a_3 such that $N[a_1,a_2] > N[a_2,a_1]$, $N[a_2,a_3] > N[a_3,a_2]$, and $N[a_3,a_1] > N[a_1,a_3]$ and such that (4.6.1) – (4.6.3) are satisfied, we get $\mathcal{E}^{new} = \{b\}$. With (4.11.1.5) and the requirement that $\mathcal{S}^{new}$ must not be empty, we get $\mathcal{S}^{new} = \{b\}$. But with (4.6.7) and $a \in \mathcal{S}^{old}$, we get $\mathcal{S}^{new} \cap \{a_1,a_2,a_3\} \neq \emptyset$. As $\mathcal{S}^{new} = \{b\}$ and $\mathcal{S}^{new} \cap \{a_1,a_2,a_3\} \neq \emptyset$ are incompatible, we get $a \notin \mathcal{S}^{old}$.
>
> Case II: Suppose $b \in \mathcal{S}^{old}$. When alternative b is replaced by alternatives b_1,b_2,b_3 such that $N[b_1,b_2] > N[b_2,b_1]$, $N[b_2,b_3] > N[b_3,b_2]$, and $N[b_3,b_1] > N[b_1,b_3]$ and such that (4.6.1) – (4.6.3) are satisfied, we get $\mathcal{E}^{new} = \{a\}$. With (4.11.1.5) and the requirement that $\mathcal{S}^{new}$ must not be empty, we get $\mathcal{S}^{new} = \{a\}$. But with (4.6.7) and $b \in \mathcal{S}^{old}$, we get $\mathcal{S}^{new} \cap \{b_1,b_2,b_3\} \neq \emptyset$. As $\mathcal{S}^{new} = \{a\}$ and $\mathcal{S}^{new} \cap \{b_1,b_2,b_3\} \neq \emptyset$ are incompatible, we get $b \notin \mathcal{S}^{old}$.
>
> However, $a \notin \mathcal{S}^{old}$ and $b \notin \mathcal{S}^{old}$ together are incompatible with the requirement that $\mathcal{S}^{old}$ must not be empty.

In short: When a weak Condorcet winner is replaced by a set of clones, as defined in (4.6.1) – (4.6.3), it is not guaranteed that at least one of these clones is a weak Condorcet winner. Therefore, (4.11.1.5) cannot be compatible with independence of clones.

The above examples demonstrate that, to satisfy reversal symmetry and independence of clones, we have, in some situations, to allow alternatives, which are not weak Condorcet winners, to be among the potential winners.

So the maximum, that we could ask for, is:

(4.11.1.6) $\quad \mathcal{E} \subseteq \mathcal{S}$.

Formulation (4.11.1.6) says that every weak Condorcet winner should be a potential winner, but it makes no stipulations about those alternatives which are not weak Condorcet winners. In (4.11.1.6), the presumption “ $\mathcal{E} \neq \emptyset$ ” is not needed. We don’t have to write “ $\mathcal{E} \neq \emptyset \Rightarrow \mathcal{E} \subseteq \mathcal{S}$ ” because the empty set is, by definition, subset of every set.

The following proof demonstrates that the Schulze method satisfies (4.11.1.6) and that, therefore, (4.11.1.6) is compatible with reversal symmetry and independence of clones.

Claim:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.1.6).

Proof:

Step 1:

(2.1.4) says that all ties have equivalent strengths. So without loss of generality, we can set

(4.11.1.7) $\forall\, x \in \mathbb{N}_0\text{: } (x,x) \approx_D (1,1).$

Step 2:

Suppose $a \in A$ is a weak Condorcet winner. Then, for every $b \in A \setminus \{a\}$, the link ab is already a path from alternative a to alternative b that contains no defeat. Therefore, with (2.1.5) and (4.11.1.7), we get

(4.11.1.8) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}\text{: } P_D[a,b] \succsim_D (N[a,b],N[b,a]) \succsim_D (1,1).$

Step 3:

Suppose $a \in A$ is a weak Condorcet winner. Suppose $b \in A \setminus \{a\}$. Suppose the link ca is the last link in the strongest path from alternative b to alternative a. As alternative a is a weak Condorcet winner, the link ca is either a tie or a defeat. Therefore, with (2.1.5) and (4.11.1.7), we get

(4.11.1.9) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}\ \exists\, c \in A \setminus \{a\}\text{: } P_D[b,a] \precsim_D (N[c,a],N[a,c]) \precsim_D (1,1).$

With (4.11.1.8) and (4.11.1.9), we get

(4.11.1.10) $\forall\, a \in \mathcal{E}\ \forall\, b \in A \setminus \{a\}\text{: } P_D[a,b] \succsim_D P_D[b,a].$

With (4.11.1.10), we get

(4.11.1.11) $a \in \mathcal{E} \Rightarrow a \in \mathcal{S}.$

With (4.11.1.11), we get (4.11.1.6). □

The following desideratum reduces the scenarios where some alternative, that is not a weak Condorcet winner, can be a potential winner:

(4.11.1.12) $\forall\ a \in \mathcal{E}\ \forall\ b \in (\ \mathcal{S} \setminus \mathcal{E}\ ){:}\ N[a,b] = N[b,a]$.

Desideratum (4.11.1.12) says that some alternative, that is not a weak Condorcet winner, can be a potential winner only when it pairwise ties all weak Condorcet winners.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.1.12).

Proof:

Suppose $a \in \mathcal{E}$ and $b \in (\ \mathcal{S} \setminus \mathcal{E}\ )$.

Step 1:

$N[a,b] < N[b,a]$ is a contradiction to the presumption that alternative a is a weak Condorcet winner.

Step 2:

It remains to be proven that $N[a,b] > N[b,a]$ is not possible.

So suppose $N[a,b] > N[b,a]$. Then ab is already a path from alternative a to alternative b that contains no tie and no defeat. Therefore, we get

(4.11.1.13) $P_D[a,b] \succsim_D (N[a,b],N[b,a])$.

Suppose the link ca is the last link in the strongest path from alternative b to alternative a. Then we get

(4.11.1.14) $P_D[b,a] \precsim_D (N[c,a],N[a,c])$.

As alternative a is a weak Condorcet winner, the link ca is either a tie or a defeat. Therefore, with (2.1.5), we get

(4.11.1.15) $(N[a,b],N[b,a]) \succ_D (N[c,a],N[a,c])$.

With (4.11.1.13), (4.11.1.14), and (4.11.1.15), we get

(4.11.1.16) $P_D[b,a] \precsim_D (N[c,a],N[a,c]) \prec_D (N[a,b],N[b,a]) \precsim_D P_D[a,b]$.

So alternative a disqualifies alternative b. But this is a contradiction to the presumption that alternative b is a potential winner. □

4.11.2. Weak Condorcet Losers

A *Condorcet loser* is an alternative $a \in A$ that loses every head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.2.1) Alternative $a \in A$ is a *Condorcet loser* $:\Leftrightarrow$
$$N[a,b] < N[b,a] \text{ for all } b \in A \setminus \{a\}.$$

A *weak Condorcet loser* is an alternative $a \in A$ that doesn't win any head-to-head contest with some other alternative $b \in A \setminus \{a\}$. In other words:

(4.11.2.2) Alternative $a \in A$ is a *weak Condorcet loser* $:\Leftrightarrow$
$$N[a,b] \leq N[b,a] \text{ for all } b \in A \setminus \{a\}.$$

Suppose $\mathcal{F}$ is the set of weak Condorcet losers. Then we get:

(4.11.2.3) $a \in \mathcal{F} :\Leftrightarrow N[a,b] \leq N[b,a]$ for all $b \in A \setminus \{a\}$.

A frequently stated desideratum says that a weak Condorcet loser should not be a potential winner. So with (4.11.2.3), we get

(4.11.2.4) $\forall\, a \in A$: ($a \in \mathcal{F} \Rightarrow a \notin \mathcal{S}$).

However, a problem with desideratum (4.11.2.4) is that it can happen that every alternative is a weak Condorcet loser. In this case, (4.11.2.4) is incompatible with the requirement that $\mathcal{S}$ must not be empty.

It can also happen that every weak Condorcet loser is, simultaneously, a weak Condorcet winner. In this case, (4.11.2.4) is incompatible with (4.11.1.6).

Example: Suppose there are only $C = 2$ alternatives $a,b \in A$. Suppose there is a pairwise tie, $N[a,b] = N[b,a]$. Then both alternatives are weak Condorcet losers and, simultaneously, weak Condorcet winners. (4.11.1.6) says: $a \in \mathcal{S}$ and $b \in \mathcal{S}$. (4.11.2.4) says: $a \notin \mathcal{S}$ and $b \notin \mathcal{S}$.

So the maximum, that we could ask for, is:

(4.11.2.5) $\forall\, a \in A$: ($a \in \mathcal{F}$ and $a \notin \mathcal{E} \Rightarrow a \notin \mathcal{S}$).

Desideratum (4.11.2.5) says that a weak Condorcet loser should not win, unless it is also a weak Condorcet winner. The following proof demonstrates that the Schulze method satisfies (4.11.2.5) and that, therefore, there is no need to weaken (4.11.2.5) any further.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.2.5).

Proof:

With $a \in \mathcal{F}$, we get

(4.11.2.6) $\forall\, b \in A \setminus \{a\}$: $N[a,b] \leq N[b,a]$.

With $a \notin \mathcal{E}$, we get

(4.11.2.7) $\exists\, b \in A \setminus \{a\}$: $N[a,b] < N[b,a]$.

When we take the alternative $b \in A \setminus \{a\}$ from (4.11.2.7), then the link ba is already a path from alternative b to alternative a that contains no tie or defeat.

Suppose the link ac is the first link in the strongest path from alternative a to alternative b. As alternative a is a weak Condorcet loser, the link ac is either a tie or a defeat. Therefore, with (2.1.5), (4.11.2.6), and (4.11.2.7), we get

(4.11.2.8) $P_D[b,a] \succsim_D (N[b,a],N[a,b]) \succ_D (N[a,c],N[c,a]) \succsim_D P_D[a,b]$.

So alternative b disqualifies alternative a. So $a \notin \mathcal{S}$. □

Another frequently stated desideratum says that a weak Condorcet loser should not be a unique winner. So with (4.11.2.3), we get

(4.11.2.9) $\forall\ a \in A$: ($a \in \mathcal{F} \Rightarrow \mathcal{S} \neq \{a\}$).

Claim:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies (4.11.2.9).

Proof:

Step 1:

(2.1.4) says that all ties have equivalent strengths. So without loss of generality, we can set

(4.11.2.10) $\forall\ x \in \mathbb{N}_0$: $(x,x) \approx_D (1,1)$.

Step 2:

Suppose $a \in A$ is a weak Condorcet loser. Then, for every $b \in A \setminus \{a\}$, the link ba is already a path from alternative b to alternative a that contains no defeat. Therefore, with (2.1.5) and (4.11.2.10), we get

(4.11.2.11) $\forall\ a \in \mathcal{F}\ \forall\ b \in A \setminus \{a\}$: $P_D[b,a] \succsim_D (N[b,a],N[a,b]) \succsim_D (1,1)$.

Step 3:

Suppose $a \in A$ is a weak Condorcet loser. Suppose $b \in A \setminus \{a\}$. Suppose the link ac is the first link in the strongest path from alternative a to alternative b. As alternative a is a weak Condorcet loser, the link ac is either a tie or a defeat. Therefore, with (2.1.5) and (4.11.2.10), we get

(4.11.2.12) $\forall\ a \in \mathcal{F}\ \forall\ b \in A \setminus \{a\}\ \exists\ c \in A \setminus \{a\}$: $P_D[a,b] \precsim_D (N[a,c],N[c,a]) \precsim_D (1,1)$.

With (4.11.2.11) and (4.11.2.12), we get

(4.11.2.13) $\forall\ a \in \mathcal{F}\ \forall\ b \in A \setminus \{a\}$: $P_D[b,a] \succsim_D P_D[a,b]$.

Step 4:

As $\mathcal{O}$ is transitive, there is an alternative d in $A \setminus \{a\}$ that is not disqualified by any other alternative in $A \setminus \{a\}$. We get

(4.11.2.14) $\exists\ d \in A \setminus \{a\}\ \forall\ e \in A \setminus \{a,d\}$: $ed \notin \mathcal{O}$.

With (4.11.2.13), we get that alternative a doesn't disqualify alternative d. With (4.11.2.14), we get that no other alternative $e \in A \setminus \{a,d\}$ disqualifies alternative d. Therefore, alternative d is a potential winner. Therefore, we get $d \in \mathcal{S}$. Therefore, we get $\mathcal{S} \neq \{a\}$. Therefore, we get (4.11.2.9). □

4.12. Increasing Sequential Independence

Increasing sequential independence says that, when alternative $a \in A$ is a winner, then there must be an alternative $d \in A \setminus \{a\}$ such that, when the used election method is applied to $A \setminus \{d\}$, then alternative a is still a winner.

The name for this criterion comes from the fact that — when the used election method satisfies this criterion and when alternative $a \in A$ is a winner and alternative $d(1) \in A \setminus \{a\}$ is an alternative such that, when the used election method is applied to $A \setminus \{d(1)\}$, then alternative a is still a winner — the same criterion can then be applied to $A \setminus \{d(1)\}$ to identify an alternative $d(2) \in A \setminus \{a,d(1)\}$ such that, when the used election method is applied to $A \setminus \{d(1),d(2)\}$, then alternative a is still a winner. When we continue applying this criterion, we get a linear order $d(1),...,d(C–1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C–1)\}$, alternative a is still a winner when the used election method is applied to $A \setminus \{d(1),...,d(i)\}$.

The motivation for this criterion is that an alternative $a \in A$ should be able to win only by disqualifying all the other alternatives directly or indirectly in some manner. It should not be possible that some alternatives $\emptyset \neq \{d(1),...,d(i)\} \subsetneq A$ disqualify each other in such a manner that the final winner comes from outside of $\{d(1),...,d(i)\}$. When increasing sequential independence is satisfied, then one alternative after the other is disqualified, so that the final winner $a \in A$ can come from outside of $\{d(1),...,d(i)\}$ only when the last remaining alternative $d(j) \in \{d(1),...,d(i)\}$ is disqualified by some alternatives outside of $\{d(1),...,d(i)\}$.

Increasing sequential independence and decreasing sequential independence (section 4.14) as criteria for single-winner election methods have been proposed by Arrow and Raynaud (1986) and generalized by Lansdowne (1996).

Definition #1:

An election method satisfies the first version of *increasing sequential independence* if the following holds:

Suppose alternative $a \in A$ is a unique winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still a unique winner.

Claim #1:

The Schulze method, as defined in section 2.2, satisfies the first version of increasing sequential independence.

Proof of claim #1:

Suppose alternative $a \in A$ is a unique winner when this election method is applied to A. Then, according to (4.1.15), alternative a disqualifies every other alternative $b \in A \setminus \{a\}$. Therefore, we get

(4.12.1) $$\forall\, b \in A \setminus \{a\}: P^{\text{old}}_{D}[a,b] \succ_D P^{\text{old}}_{D}[b,a].$$

Suppose $pred^{\text{old}}[a,x]$ is the predecessor of alternative $x \in A \setminus \{a\}$ in the strongest path from alternative a to alternative x, as calculated in section 2.3. Then a *leaf* is an alternative $y \in A \setminus \{a\}$ such that there is no alternative $x \in A \setminus \{a\}$ with $pred^{\text{old}}[a,x] = y$. As the strongest paths from alternative a to every other alternative $x \in A \setminus \{a\}$, as calculated by the Floyd-Warshall algorithm, form an arborescence, there must be at least one leaf. Alternative d is chosen arbitrarily from these leaves.

Suppose alternative d is removed. As alternative d is a leaf, alternative d is not in the strongest path from alternative a to any other alternative $b \in A \setminus \{a,d\}$. Therefore, we get

(4.12.2) $\forall\ b \in A \setminus \{a,d\}\colon P_D^{\text{new}}[a,b] \approx_D P_D^{\text{old}}[a,b].$

On the other side, when an alternative is removed, then the strengths of the strongest paths can only decrease. Therefore, we get

(4.12.3) $\forall\ b \in A \setminus \{a,d\}\colon P_D^{\text{new}}[b,a] \precsim_D P_D^{\text{old}}[b,a].$

With (4.12.2), (4.12.1), and (4.12.3), we get

(4.12.4) $\forall\ b \in A \setminus \{a,d\}\colon P_D^{\text{new}}[a,b] \approx_D P_D^{\text{old}}[a,b] \succ_D P_D^{\text{old}}[b,a] \succsim_D P_D^{\text{new}}[b,a]$

so that alternative a is still a unique winner when alternative d is removed. □

Definition #2:

An election method satisfies the second version of *increasing sequential independence* if the following holds:

Suppose alternative $a \in A$ is a potential winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still a potential winner.

Claim #2:

The Schulze method, as defined in section 2.2, satisfies the second version of increasing sequential independence.

Proof of claim #2:

Suppose alternative $a \in A$ is a potential winner when this election method is applied to A. Then, we get

(4.12.5) $\forall\ b \in A \setminus \{a\}\colon P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{old}}[b,a].$

The rest of this proof is identical to the proof of claim #1. □

4.13. *k*-Consistency

The Condorcet criterion says that, when some candidate $a \in A$ wins every head-to-head contest, then this candidate a should also be the overall winner (Condorcet, 1785).

However, many countries have a strong 3-party, 4-party or 5-party system where no single party can win a majority and where every party is willing to coalesce with every other party. In such a scenario, it seems to be rather uninteresting which candidate might win in a head-to-head contest. It is more interesting to ask whether there is some candidate who wins regardless of which candidates are nominated by the other parties.

So for example in the 3-party case with party α, party β, and party γ, it might be more interesting to ask whether there is a candidate from party α who wins every 3-way contest between himself and a candidate from party β and a candidate from party γ. If there is such a candidate, then this candidate should also be the overall winner.

More generally, if there is a $k \in \mathbb{N}$ with $k \geq 2$ such that there is an alternative $a \in A$ such that alternative a wins every k-way contest, then alternative a should also be the overall winner. This criterion is called *k-set-consistency* (Heitzig, 2004) or *k-consistency* (Simmons, 2004).

k-consistency as a criterion for single-winner election methods has been proposed by Heitzig (2004) and Simmons (2004). However, a similar idea had already been formulated by Saari (Saari, 2001, pages 154–156; Lagerspetz, 2015, page 207). To question the relevance of the Condorcet criterion, Saari argued that it could happen that some alternative $a \in A$ wins every 2-way contest, some other alternative $b \in A \setminus \{a\}$ wins every 3-way contest, some other alternative $c \in A \setminus \{a,b\}$ wins every 4-way contest, etc., so that, with the same justification, every alternative could claim to be the overall winner. However, the fact that the Schulze method satisfies k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$ means that there are election methods where it is impossible to create examples such that there are $m,n \in \mathbb{N}$ with $2 \leq m < n \leq C$ such that some alternative $a \in A$ wins every m-way contest and some other alternative $b \in A \setminus \{a\}$ wins every n-way contest. So for these election methods, Saari’s scenario is not possible, so that his criticism of the Condorcet criterion doesn’t work.

There are five different versions for *k*-consistency.

The first version addresses unique winners. This version says that, when alternative $a \in A$ is a unique winner in every *k*-way contest, then alternative *a* should also be a unique winner overall. For $k = 2$, the first version of *k*-consistency is identical to the Condorcet criterion (section 4.7).

The second version addresses potential winners. This version says that, when alternative $a \in A$ is a potential winner in every *k*-way contest, then alternative *a* should also be a potential winner overall. For $k = 2$, the second version of *k*-consistency is identical to the desideratum that weak Condorcet winners should always be potential winners; equation (4.11.1.6).

The third version addresses the set of potential winners. This version says that, when in every *k*-way contest (that contains at least one alternative of the set $\emptyset \neq B \subsetneq A$) the winner comes from the set *B*, then the winner must also come from the set *B* when the method is applied to *A*. For $k = 2$, the third version of *k*-consistency is identical to the Smith criterion (section 4.7).

The fourth version says that, when alternative $a \in A$ is not a unique winner in any *k*-way contest, then alternative *a* should also be not a unique winner overall. For $k = 2$, the fourth version of *k*-consistency is identical to the desideratum that a weak Condorcet loser should not be a unique winner; equation (4.11.2.9).

The fifth version says that, when alternative $a \in A$ is not a potential winner in any *k*-way contest, then alternative *a* should also be not a potential winner overall. For $k = 2$, the fifth version of *k*-consistency is identical to the Condorcet loser criterion (section 4.7).

4.13.1. Formulation #1

Definition:

> Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the first version of *k-consistency* if the following holds:
>
> > Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is a unique winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also a unique winner when this election method is applied to A.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the first version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when alternative $a \in A$ is not a unique winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is not a unique winner.

Proof (details):

Suppose alternative $a \in A$ is not a unique winner when the Schulze method is applied to A. Then there must be an alternative $b \in A \setminus \{a\}$ with

(4.13.1.1) $\quad P_D[b,a] \succsim_D P_D[a,b]$.

We set

(4.13.1.2) $\quad (z_1,z_2) := P_D[b,a]$

to stress that this value is constant for the rest of this proof.

Suppose $c(1),\ldots,c(n)$ is the strongest path from alternative $b \equiv c(1)$ to alternative $a \equiv c(n)$. Then we get

(4.13.1.3) $\quad \forall\, i = 1,\ldots,(n-1)$: $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

Especially, we get

(4.13.1.4) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

When there is more than one path from alternative b to alternative a of strength (z_1,z_2) then, without loss of generality, we take the shortest of these paths (in terms of its number of links). Therefore, we get

(4.13.1.5) $\quad \forall\, i,j \in \{1,\ldots,n\}$ with $j - i \geq 2$: $(N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

Otherwise, if there was a link $c(i),c(j)$ with $(N[c(i),c(j)],N[c(j),c(i)]) \succsim_D (z_1,z_2)$ and $j - i \geq 2$, then we could find a shorter path of strength (z_1,z_2) by omitting the alternatives $c(i+1),...,c(j–1)$. This would be a contradiction to the presumption that $c(1),...,c(n)$ is the shortest path of strength (z_1,z_2).

With (2.1.5), we get that every path that contains no defeat is always stronger than every path that contains a defeat.

It is easy to prove that, for every pair of alternatives $x,y \in A$, there is a path from alternative x to alternative y that contains no defeat or a path from alternative y to alternative x that contains no defeat. To prove this, we only have to consider the links xy and yx because the link xy is already a path from alternative x to alternative y and the link yx is already a path from alternative y to alternative x. If $N[x,y] > N[y,x]$, then the link xy is a path from alternative x to alternative y that contains no defeat. If $N[x,y] < N[y,x]$, then the link yx is a path from alternative y to alternative x that contains no defeat. If $N[x,y] = N[y,x]$, then the link xy is a path from alternative x to alternative y that contains no defeat and the link yx is a path from alternative y to alternative x that contains no defeat.

With (4.13.1.1) and the above considerations, we get that the path $c(1),...,c(n)$ contains no defeat. {Otherwise: Suppose the path $c(1),...,c(n)$ contains a defeat. Then [as, for every pair of alternatives $x,y \in A$, there is a path from alternative x to alternative y that contains no defeat or a path from alternative y to alternative x that contains no defeat] there must be a path $d(1),...,d(r)$ from alternative b to alternative a that contains no defeat or a path $e(1),...,e(s)$ from alternative a to alternative b that contains no defeat. If there is a path $d(1),...,d(r)$ from alternative b to alternative a that contains no defeat then, according to (2.1.5), this path is stronger than the path $c(1),...,c(n)$ that contains a defeat; this is a contradiction to the presumption that the path $c(1),...,c(n)$ is the strongest path from alternative b to alternative a. If there is no path from alternative b to alternative a that contains no defeat, but a path $e(1),...,e(s)$ from alternative a to alternative b that contains no defeat then, according to (2.1.5), this path is stronger than the path $c(1),...,c(n)$ that contains a defeat; this is a contradiction to (4.13.1.1).} Especially, the link $c(n–1),c(n)$ is not a defeat. Therefore, we get

(4.13.1.6) $\forall\ i = 1,...,(n–1){:}\ N[c(i),c(i+1)] \geq N[c(i+1),c(i)]$.

Especially, we get

(4.13.1.7) $N[c(n–1),c(n)] \geq N[c(n),c(n–1)]$.

With (2.1.5) and (4.13.1.7), we get

(4.13.1.8) $(N[c(n–1),c(n)],N[c(n),c(n–1)]) \gtrsim_D (N[c(n),c(n–1)],N[c(n–1),c(n)])$.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.5), then the first version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the Condorcet criterion. However, it has already been proven in section 4.7 that the Schulze method satisfies the Condorcet criterion when $\succ_D$ satisfies (2.1.5).

Case #2: $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k–2),c(n–1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n–1)$ to $c(n)$ of at least $(N[c(n–1),c(n)],N[c(n),c(n–1)]) \approx_D (z_1,z_2)$ because, according to (4.13.1.4), already the link $c(n–1),c(n)$ is a path from $c(n–1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n–1)$ of more than $(N[c(n–1),c(n)],N[c(n),c(n–1)])$ because, according to (4.13.1.5), every link from $c(1)$, ..., $c(k–2)$ to $c(n–1)$ is weaker than (z_1,z_2) and, according to (4.13.1.8), the link $c(n),c(n–1)$ is not stronger than $(N[c(n–1),c(n)],N[c(n),c(n–1)])$.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(n–1)$. So either alternative $c(n–1)$ is also a potential winner or, according to (4.1.14), alternative $c(n–1)$ must be disqualified by some other potential winner. In both cases, alternative $c(n)$ is not a unique winner.

Case #3: $k \geq n$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k–n$ additional alternatives from A.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.1.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of more than (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(1)$. So either alternative $c(1)$ is also a potential winner or, according to (4.1.14), alternative $c(1)$ must be disqualified by some other potential winner. In both cases, alternative $c(n)$ is not a unique winner. □

4.13.2. Formulation #2

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the second version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is a potential winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$. Then alternative a is also a potential winner when this election method is applied to A.

Claim:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the second version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when alternative $a \in A$ is not a potential winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is not a potential winner.

Proof (details):

Suppose alternative $a \in A$ is not a potential winner when the Schulze method is applied to A. Then there must be an alternative $b \in A \setminus \{a\}$ with

(4.13.2.1) $\quad P_D[b,a] \succ_D P_D[a,b]$.

We set

(4.13.2.2) $\quad (z_1,z_2) := P_D[b,a]$

to stress that this value is constant for the rest of this proof.

Suppose $c(1),...,c(n)$ is the strongest path from alternative $b \equiv c(1)$ to alternative $a \equiv c(n)$. Then we get

(4.13.2.3) $\quad \forall\, i = 1,...,(n-1)$: $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

Especially, we get

(4.13.2.4) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

With the same arguments as for (4.13.1.5), we get

(4.13.2.5) $\quad \forall\, i,j \in \{1,...,n\}$ with $j - i \geq 2$: $(N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

With (2.1.4) and (2.1.5), we get that every path that contains no defeat or tie is always stronger than every path that contains a defeat or tie.

It is easy to prove that the path $c(1),...,c(n)$ contains no defeat or tie. Therefore, we get

(4.13.2.6) $\forall\ i = 1,...,(n–1)$: $N[c(i),c(i+1)] > N[c(i+1),c(i)]$.

Especially, we get

(4.13.2.7) $N[c(n–1),c(n)] > N[c(n),c(n–1)]$.

With (2.1.5) and (4.13.2.7), we get

(4.13.2.8) $(N[c(n–1),c(n)],N[c(n),c(n–1)]) \succ_D (N[c(n),c(n–1)],N[c(n–1),c(n)])$.

Proof for (4.13.2.6):

It has already been shown in the proof in section 4.13.1 that, when $\succ_D$ satisfies (2.1.5), then the path $c(1),...,c(n)$ contains no defeat. So it remains to be proven that the path $c(1),...,c(n)$ contains no tie.

To prove that the path $c(1),...,c(n)$ contains no tie, we presume that (2.1.4), (2.1.5), and (4.13.2.1) are satisfied and that the path $c(1),...,c(n)$ contains a tie and then we will show that this leads to a contradiction.

(2.1.4) says that all ties have equivalent strengths. (2.1.5) says that every win is stronger than every tie. So when the path $c(1),...,c(n)$ contains no defeat, but at least one tie then, without loss of generality, we can set

(4.13.2.9) $P_D[b,a] \approx_D (1,1)$.

To get to a contradiction, it is sufficient to consider the link *ab*.

Case #A: If the link *ab* is a win (i.e. $N[a,b] > N[b,a]$) or a tie (i.e. $N[a,b] = N[b,a]$), then this link is already a path from alternative *a* to alternative *b* that contains no defeat. Therefore, with (2.1.4), (2.1.5), and (4.13.2.9), we get $P_D[a,b] \succsim_D (N[a,b],N[b,a]) \succsim_D (1,1) \approx_D P_D[b,a]$. But this is a contradiction to (4.13.2.1).

Case #B: If the link *ab* is a defeat (i.e. $N[a,b] < N[b,a]$), then the link *ba* is a path from alternative *b* to alternative *a* that contains no defeat or tie. But then, according to (2.1.5), the link *ba* is stronger than the path $c(1),...,c(n)$ that contains a tie. But this is a contradiction to the presumption that the path $c(1),...,c(n)$ is the strongest path from alternative *b* to alternative *a*.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.4) and (2.1.5), then the second version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the desideratum that a weak Condorcet winner is always a potential winner. However, it has already been proven in section 4.11 that the Schulze method satisfies this desideratum when $\succ_D$ satisfies (2.1.4) and (2.1.5).

Case #2: $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k–2),c(n–1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n–1)$ to $c(n)$ of at least $(N[c(n–1),c(n)],N[c(n),c(n–1)]) \succsim_D (z_1,z_2)$ because, according to (4.13.2.4), already the link $c(n–1),c(n)$ is a path from $c(n–1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n–1)$ of at least $(N[c(n–1),c(n)],N[c(n),c(n–1)])$ because, according to (4.13.2.5), every link from $c(1)$, ..., $c(k–2)$ to $c(n–1)$ is weaker than (z_1,z_2) and, according to (4.13.2.8), the link $c(n),c(n–1)$ is weaker than $(N[c(n–1),c(n)],N[c(n),c(n–1)])$.

Therefore, alternative $c(n–1)$ disqualifies alternative $c(n)$, so that alternative $c(n)$ is not a potential winner.

Case #3: $k \geq n$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k–n$ additional alternatives from A.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.2.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of at least (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(1)$ disqualifies alternative $c(n)$, so that alternative $c(n)$ is not a potential winner. □

4.13.3. Formulation #3

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the third version of *k-consistency* if the following holds:

Suppose $C \geq k$ is the number of alternatives in A. Suppose $\mathcal{S}|_{\tilde{A}}$ is the set of potential winners when this election method is applied to $\emptyset \neq \tilde{A} \subseteq A$. Suppose $\emptyset \neq B \subsetneq A$. Suppose $\mathcal{S}|_{\tilde{A}} \subseteq B$ whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ and $B \cap \tilde{A} \neq \emptyset$. Then we must also get $\mathcal{S}|_A \subseteq B$. In short:

$\forall\ \emptyset \neq B \subsetneq A$: (($\forall\ \tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ and $B \cap \tilde{A} \neq \emptyset$: $\mathcal{S}|_{\tilde{A}} \subseteq B$) $\Rightarrow$ ($\mathcal{S}|_A \subseteq B$)).

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the third version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Proof (overview):

We will show how, when $\mathcal{S}|_A \not\subseteq B$, we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ and $B \cap \tilde{A} \neq \emptyset$ such that, when the Schulze method is applied to $\tilde{A}$, we get $\mathcal{S}|_{\tilde{A}} \not\subseteq B$.

Proof (details):

Suppose $r := |\ B\ |$ is the number of alternatives in B. With $\emptyset \neq B \subsetneq A$, we get: $0 < r < C$.

Suppose $\mathcal{S}|_A \not\subseteq B$. Then there must be an alternative $b \in A$ with $b \in \mathcal{S}|_A$ and $b \notin B$. With $b \in \mathcal{S}|_A$ we get

(4.13.3.1) $\forall\ a \in A \setminus \{b\}$: $P_D[b,a] \succsim_D P_D[a,b]$.

Case #1: $k = 2$.

When $\succ_D$ satisfies (2.1.5), then the third version of 2-consistency, applied to the Schulze method, means that the Schulze method should satisfy the Smith criterion. However, it has already been proven in section 4.7 that the Schulze method satisfies the Smith criterion when $\succ_D$ satisfies (2.1.5).

Case #2: $k > C - r$.

In section 4.12, we have proven that, when alternative $b \in A$ is a potential winner, then there is a linear order $d(1),...,d(C-1)$ of the alternatives in $A \setminus \{b\}$, such that, when the Schulze method is applied to $A \setminus \{d(1),...,d(C-k)\}$, then alternative b is still a potential winner.

As $k > C - r$, every set $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ contains at least $k + r - C \geq 1$ alternatives of B. Therefore, we get $B \cap \tilde{A} \neq \varnothing$ for every set $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$. Therefore, we can choose $\tilde{A} :=$ $A \setminus \{d(1),...,d(C-k)\}$.

Case #3: $3 \leq k \leq C - r$.

We take some $b \in A$ with $b \in \mathcal{S}|_A$ and $b \notin B$. We sort the alternatives $\{a(1),...,a(C-1)\}$ in $A \setminus \{b\}$ such that

$\forall\ i,j \in \mathbb{N}$ with $1 \leq i < C$ and $1 \leq j < C$: ($pred[b,a(j)] = a(i) \Rightarrow i < j$).

Suppose $y \in \mathbb{N}$ with $1 \leq y < C$ is the smallest number with $a(y) \in B$. Then we get $a(x) \notin B$ for all $x \in \mathbb{N}$ with $1 \leq x < y$. Furthermore, when $d(1),...,d(m)$ is the strongest path from alternative $b \equiv d(1)$ to alternative $a(y) \equiv d(m)$ then, with the definition for $pred[i,j]$ and with the definition for the order of $\{a(1),...,a(C-1)\}$, we get $\{d(1),...,d(m-1)\} \subseteq \{b,a(1),...,a(y-1)\} \subseteq A \setminus B$.

We set

(4.13.3.2) $\quad (z_1,z_2) := P_D[b,a(y)]$

to stress that this value is constant for the rest of this proof.

We now shorten the path $d(1),...,d(m)$ by removing possible short cuts. So when there is a link $d(i),d(j)$ with $(N[d(i),d(j)],N[d(j),d(i)]) \succsim_D (z_1,z_2)$ and $j - i \geq 2$, we remove the alternatives $d(i+1),...,d(j-1)$ from this path. We continue removing possible short cuts, until the resulting path contains no short cuts anymore. The resulting path will be called $c(1),...,c(n)$.

We get $c(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < n$, because we have already established $d(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < m$ and because, when we shortened the path $d(1),...,d(m)$, we only removed and didn't add alternatives.

With the same arguments as for (4.13.1.3) – (4.13.1.8), we get (4.13.3.3) – (4.13.3.8):

(4.13.3.3) $\quad \forall\ i = 1,...,(n-1)$: $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \succsim_D (z_1,z_2)$.

(4.13.3.4) $\quad (N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$.

(4.13.3.5) $\quad \forall\ i,j \in \{1,...,n\}$ with $j - i \geq 2$: $(N[c(i),c(j)],N[c(j),c(i)]) \prec_D (z_1,z_2)$.

(4.13.3.6) $\quad \forall\ i = 1,...,(n-1)$: $N[c(i),c(i+1)] \geq N[c(i+1),c(i)]$.

(4.13.3.7) $N[c(n-1),c(n)] \geq N[c(n),c(n-1)]$.

(4.13.3.8) $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (N[c(n),c(n-1)],N[c(n-1),c(n)])$.

With the above considerations, we can now show how the subset $\tilde{A} \subseteq A$ can be chosen.

Case #3a: $3 \leq k < n$.

Here, we choose $\tilde{A} := \{c(1),...,c(k-2),c(n-1),c(n)\}$.

When the Schulze method is applied to $\tilde{A}$, then there is a path from $c(n-1)$ to $c(n)$ of at least $(N[c(n-1),c(n)],N[c(n),c(n-1)]) \succsim_D (z_1,z_2)$ because, according to (4.13.3.4), already the link $c(n-1),c(n)$ is a path from $c(n-1)$ to $c(n)$ of this strength.

On the other side, there cannot be a path in $\tilde{A}$ from $c(n)$ to $c(n-1)$ of more than $(N[c(n-1),c(n)],N[c(n),c(n-1)])$ because, according to (4.13.3.5), every link from $c(1)$, ..., $c(k-2)$ to $c(n-1)$ is weaker than (z_1,z_2) and, according to (4.13.3.8), the link $c(n),c(n-1)$ is not stronger than $(N[c(n-1),c(n)],N[c(n),c(n-1)])$.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(n-1)$. So either alternative $c(n-1)$ is also a potential winner or, according to (4.1.14), alternative $c(n-1)$ must be disqualified by some other potential winner in $\tilde{A}$. As $c(i) \notin B$ for all $i \in \mathbb{N}$ with $1 \leq i < n$, this potential winner comes from outside B.

Case #3b: $n \leq k \leq C - r$.

Here, $\tilde{A}$ consists of the alternatives $c(1),...,c(n)$ and $k-n$ additional alternatives from $A \setminus B$.

As $\{c(1),...,c(n)\} \subseteq \tilde{A}$, there is a path in $\tilde{A}$ from alternative $c(1)$ to alternative $c(n)$ of strength (z_1,z_2). On the other side, we get, with (4.13.3.1), that there cannot be a path in $\tilde{A}$ from alternative $c(n)$ to alternative $c(1)$ of more than (z_1,z_2) because, when alternatives are removed from A, then the strength of the strongest path from alternative $c(n)$ to alternative $c(1)$ can only decrease.

Therefore, alternative $c(n)$ cannot disqualify alternative $c(1)$. So either alternative $c(1)$ is also a potential winner or, according to (4.1.14), alternative $c(1)$ must be disqualified by some other potential winner in $\tilde{A}$. As $e \notin B$ for all $e \in \tilde{A} \setminus \{c(n)\}$, this potential winner comes from outside B. □

4.13.4. Formulation #4

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the fourth version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is not a unique winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $| \tilde{A} | = k$ and $a \in \tilde{A}$. Then alternative a is also not a unique winner when this election method is applied to A.

Claim:

The Schulze method, as defined in section 2.2, satisfies the fourth version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Remark:

Presumptions (2.1.4) and (2.1.5) are not needed in the following proof. However, only when $\succ_D$ satisfies (2.1.4) and (2.1.5), the fourth version of k-consistency with $k = 2$ is identical to the desideratum that a weak Condorcet loser should not be a unique winner.

Proof (overview):

We will show how, when alternative $a \in A$ is a unique winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $| \tilde{A} | = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is a unique winner.

Proof (details):

In section 4.12, we have proven that, when alternative $a \in A$ is a unique winner, then there is a linear order $d(1),...,d(C\text{–}1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C\text{–}1)\}$, alternative a is still a unique winner when the Schulze method is applied to $A \setminus \{d(1),...,d(i)\}$.

Therefore, for $k \in \mathbb{N}$ with $2 \leq k \leq C$, we can simply choose $\tilde{A} := A \setminus \{d(1),...,d(C\text{–}k)\}$. □

4.13.5. Formulation #5

Definition:

Suppose $k \in \mathbb{N}$ with $k \geq 2$. An election method satisfies the fifth version of *k-consistency* if the following holds:

> Suppose $C \geq k$ is the number of alternatives in A. Suppose alternative $a \in A$ is not a potential winner whenever this election method is applied to some subset $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ and $a \in \tilde{A}$. Then alternative a is also not a potential winner when this election method is applied to A.

Claim:

The Schulze method, as defined in section 2.2, satisfies the fifth version of k-consistency for every $k \in \mathbb{N}$ with $k \geq 2$.

Remark:

Presumption (2.1.5) is not needed in the following proof. However, only when $\succ_D$ satisfies (2.1.5), the fifth version of k-consistency with $k = 2$ is identical to the Condorcet loser criterion.

Proof (overview):

We will show how, when alternative $a \in A$ is a potential winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\ \tilde{A}\ | = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is a potential winner.

Proof (details):

In section 4.12, we have proven that, when alternative $a \in A$ is a potential winner, then there is a linear order $d(1),...,d(C-1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C-1)\}$, alternative a is still a potential winner when the Schulze method is applied to $A \setminus \{d(1),...,d(i)\}$.

Therefore, for $k \in \mathbb{N}$ with $2 \leq k \leq C$, we can simply choose $\tilde{A} := A \setminus \{d(1),...,d(C-k)\}$. □

4.14. Decreasing Sequential Independence

Decreasing sequential independence says that, when alternative $a \in A$ is not a winner, then there must be an alternative $d \in A \setminus \{a\}$ such that, when the used election method is applied to $A \setminus \{d\}$, then alternative a is still not a winner.

The name for this criterion comes from the fact that — when the used election method satisfies this criterion and when alternative $a \in A$ is not a winner and alternative $d(1) \in A \setminus \{a\}$ is an alternative such that, when the used election method is applied to $A \setminus \{d(1)\}$, then alternative a is still not a winner — the same criterion can then be applied to $A \setminus \{d(1)\}$ to identify an alternative $d(2) \in A \setminus \{a,d(1)\}$ such that, when the used election method is applied to $A \setminus \{d(1),d(2)\}$, then alternative a is still not a winner. When we continue applying this criterion, we get a linear order $d(1),...,d(C–1)$ of the alternatives in $A \setminus \{a\}$ such that, for every $i \in \{1,...,(C–1)\}$, alternative a is still not a winner when the used election method is applied to $A \setminus \{d(1),...,d(i)\}$.

Increasing sequential independence and decreasing sequential independence address opposite problems. On the one side, *increasing sequential independence* says that it should not be possible that alternatives $\emptyset \neq \{d(1),...,d(i)\} \subsetneq A$ *harm* each other in such a manner that the final winner comes from outside of $\{d(1),...,d(i)\}$. On the other side, *decreasing sequential independence* says that, when no proper subset of $\{d(1),...,d(i)\}$ can disqualify every alternative outside of $\{d(1),...,d(i)\}$, then the alternatives $\{d(1),...,d(i)\}$ should not *help* each other in such a manner that $\{d(1),...,d(i)\}$ together disqualify every alternative outside of $\{d(1),...,d(i)\}$.

The fact that the Schulze method satisfies decreasing sequential independence follows directly from the fact that the Schulze method satisfies the first and the second version of k-consistency for every $k \in \mathbb{N}$ with $2 \leq k \leq C$ (sections 4.13.1 and 4.13.2).

Definition #1:

An election method satisfies the first version of *decreasing sequential independence* if the following holds:

Suppose there are at least $C \geq 3$ alternatives. Suppose alternative $a \in A$ is not a unique winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still not a unique winner.

Claim #1:

If $\succ_D$ satisfies (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the first version of decreasing sequential independence.

Proof of claim #1:

Suppose alternative $a \in A$ is not a unique winner when this election method is applied to A. In section 4.13.1, we have shown that, when alternative $a \in A$ is not a unique winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is still not a unique winner. When we choose $k = C$–1, we get the first version of decreasing sequential independence. □

Definition #2:

An election method satisfies the second version of *decreasing sequential independence* if the following holds:

Suppose there are at least $C \geq 3$ alternatives. Suppose alternative $a \in A$ is not a potential winner when this election method is applied to A. Then there must be a (not necessarily unique) alternative $d \in A \setminus \{a\}$ such that, when this election method is applied to $A \setminus \{d\}$, then alternative a is still not a potential winner.

Claim #2:

If $\succ_D$ satisfies (2.1.4) and (2.1.5), then the Schulze method, as defined in section 2.2, satisfies the second version of decreasing sequential independence.

Proof of claim #2:

Suppose alternative $a \in A$ is not a potential winner when this election method is applied to A. In section 4.13.2, we have shown that, when alternative $a \in A$ is not a potential winner (when this election method is applied to A), we can create, for every $k \in \mathbb{N}$ with $2 \leq k \leq C$, a subset $\tilde{A} \subseteq A$ with $|\tilde{A}| = k$ and $a \in \tilde{A}$ such that, when the Schulze method is applied to $\tilde{A}$, alternative a is still not a potential winner. When we choose $k = C$–1, we get the second version of decreasing sequential independence. □

4.15. Weak Independence from Pareto-Dominated Alternatives

Suppose an alternative j is added such that:

(3.6.1) $\exists\, i \in A^{\text{old}}\ \forall\, v \in V$: $i \succsim_v^{\text{new}} j$.

(3.6.2) $\forall\, g,h \in A^{\text{old}}\ \forall\, v \in V$: $g \succ_v^{\text{old}} h \Leftrightarrow g \succ_v^{\text{new}} h$.

Then *independence from Pareto-dominated alternatives* (IPDA) says that we must get:

(3.6.3) $\forall\, g,h \in A^{\text{old}}$: $gh \in \mathcal{O}^{\text{old}} \Leftrightarrow gh \in \mathcal{O}^{\text{new}}$.

(3.6.4) $\forall\, g \in A^{\text{old}}$: $g \in \mathcal{S}^{\text{old}} \Leftrightarrow g \in \mathcal{S}^{\text{new}}$.

In example 6 (section 3.6) and example 7 (section 3.7), we have seen that the Schulze method violates IPDA. In example 6, the winner is changed from alternative $a \in A^{\text{old}}$ to alternative $b \in A^{\text{old}} \setminus \{a\}$ by adding an alternative e with

(4.15.1) $\exists\, d \in A^{\text{old}} \setminus \{a,b\}\ \forall\, v \in V$: $d \succsim_v^{\text{new}} e$.

In example 7, the winner is changed from alternative $a \in A^{\text{old}}$ to alternative $b \in A^{\text{old}} \setminus \{a\}$ by adding an alternative e with

(4.15.2) $\forall\, v \in V$: $a \succsim_v^{\text{new}} e$.

It has already been mentioned in section 4.8 that IPDA and (4.8.5) are incompatible. In example 6(old), we have $\mathfrak{B}_D^{\text{old}} = \{a, c, d\}$. In example 6(new), we have $\mathfrak{B}_D^{\text{new}} = \{b\}$. Therefore, $\mathfrak{B}_D^{\text{old}} \cap \mathfrak{B}_D^{\text{new}} = \emptyset$. So (4.8.5) says that the winner must change. In example 7(old), we have $\mathfrak{B}_D^{\text{old}} = \{a, c, d\}$. In example 7(new), we have $\mathfrak{B}_D^{\text{new}} = \{b\}$ so that, again, (4.8.5) says that the winner must change.

So we cannot exclude that the winner is changed from alternative $a \in A^{\text{old}}$ to alternative $b \in A^{\text{old}} \setminus \{a\}$ by adding an alternative e with (4.15.1) or (4.15.2). But we will prove that the winner cannot be changed by adding an alternative e with

(4.15.3) $\forall\, v \in V$: $b \succsim_v^{\text{new}} e$.

Definition:

An election method satisfies *weak independence from Pareto-dominated alternatives* (wIPDA) if the following holds:

Suppose $b \notin \mathcal{S}^{\text{old}}$.

Suppose an alternative e is added with (3.6.2) and

(4.15.4) $\forall\, v \in V$: $b \succsim_v^{\text{new}} e$.

Then we get: $b \notin \mathcal{S}^{\text{new}}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies *weak independence from Pareto-dominated alternatives*.

Proof:

Suppose $b \notin \mathcal{S}^{\text{old}}$. Then, there was an alternative $a \in A^{\text{old}} \setminus \{b\}$ with $ab \in \mathcal{O}^{\text{old}}$. With $ab \in \mathcal{O}^{\text{old}}$, we get

(4.15.5) $$P_D^{\text{old}}[a,b] \succ_D P_D^{\text{old}}[b,a].$$

Suppose an alternative e is added with (3.6.2) and (4.15.4).

Suppose $c(1),...,c(n)$ was the strongest path from alternative a to alternative b in A^{old}. Then $c(1),...,c(n)$ is still a path from alternative a to alternative b in A^{new} of the same strength. Therefore, we get

(4.15.6) $$P_D^{\text{new}}[a,b] \succsim_D P_D^{\text{old}}[a,b].$$

Suppose $d(1),...,d(m)$ is the strongest path from alternative b to alternative a in A^{new}.

Case I: Suppose $d(1),...,d(m)$ does not contain alternative e. Then $d(1),...,d(m)$ was a path from alternative b to alternative a in A^{old} with the same strength. Therefore, we get: $P_D^{\text{old}}[b,a] \succsim_D P_D^{\text{new}}[b,a]$.

Case II: Suppose $d(1),...,d(m)$ contains alternative e. Suppose $d(s)$ is the last occurrence of alternative e in the path $d(1),...,d(m)$. With (2.1.1), (4.15.4), and $d(s) \equiv e$, we get: $(N[b,d(s+1)],N[d(s+1),b]) \succsim_D (N[d(s),d(s+1)],N[d(s+1),d(s)])$. So $b,d(s+1),...,d(m)$ was a path from alternative b to alternative a in A^{old} of at least the same strength as $d(1),...,d(m)$. Therefore, we get: $P_D^{\text{old}}[b,a] \succsim_D P_D^{\text{new}}[b,a]$.

So, with Case I and Case II, we get

(4.15.7) $$P_D^{\text{old}}[b,a] \succsim_D P_D^{\text{new}}[b,a].$$

With (4.15.6), (4.15.5), and (4.15.7), we get

(4.15.8) $$P_D^{\text{new}}[a,b] \succsim_D P_D^{\text{old}}[a,b] \succ_D P_D^{\text{old}}[b,a] \succsim_D P_D^{\text{new}}[b,a].$$

With (4.15.8), we get $ab \in \mathcal{O}^{\text{new}}$ and, therefore, $b \notin \mathcal{S}^{\text{new}}$. □

5. Tie-Breaking

It can happen that the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* are the same link, say *cd*. In this case, the Schulze method is indifferent between alternative *a* and alternative *b*, i.e. $ab \notin \mathcal{O}$ and $ba \notin \mathcal{O}$. See sections 3.3, 3.8, 3.9, and 4.2.

In this section, we recommend that, to resolve this indifference, the link *cd* should be declared *forbidden* and the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a*, that don’t contain *forbidden* links, should be calculated. Either this indifference is now resolved or, again, the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* are the same link, say *ef*. In the latter case, the link *ef* is declared *forbidden* and the strongest paths that don’t contain *forbidden* links are calculated. This procedure is repeated until this indifference is resolved.

The resulting Schulze relation will be called $\mathcal{O}_{final}$. The resulting set of potential winners will be called $\mathcal{S}_{final}$. The precise definitions for $\mathcal{O}_{final}$ and $\mathcal{S}_{final}$ will be given in (5.1.2) and (5.1.3).

In example 3 (section 3.3), the link *cd* is the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a*. Therefore, the link *cd* is declared *forbidden*. The strongest path from alternative *a* to alternative *b*, that doesn’t contain *forbidden* links, is *a*,(33,30),*b*. The strongest path from alternative *b* to alternative *a*, that doesn’t contain *forbidden* links, is *b*,(30,33),*a*. Therefore, we get $ab \in \mathcal{O}_{final}$.

5.1. Calculating a Complete Ranking Using a Tie-Breaking Ranking of the Links

Suppose $\mathcal{LO}_{A \times A}$ is the set of linear orders on $A \times A$. Then a *Tie-Breaking Ranking of the Links* (TBRL) is a linear order $\sigma \in \mathcal{LO}_{A \times A}$ with the following property:

(5.1.1) $$(N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m]) \Rightarrow ij \succ_\sigma mn.$$

Suppose $\sigma \in \mathcal{LO}_{A \times A}$ is a linear order on $A \times A$ with property (5.1.1). Then we calculate $\mathcal{O}_{final}(\sigma)$ and $\mathcal{S}_{final}(\sigma)$ as described in stages 1–4:

Stage 1 (initialization):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            Pσ[i,j] : = ij
        end
    end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            for k : = 1 to C
            begin
                if ( i ≠ k ) then
                begin
                    if ( j ≠ k ) then
                    begin
                        if ( P_σ[j,k] ≺_σ min_σ { P_σ[j,i], P_σ[i,k] } ) then
                        begin
                            P_σ[j,k] : = min_σ { P_σ[j,i], P_σ[i,k] }
                        end
                    end
                end
            end
        end
    end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the set of potential winners):

```
O_final(σ) : = ∅
S_final(σ) : = A
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            if ( P_σ[j,i] ≻_σ P_σ[i,j] ) then
            begin
                O_final(σ) : = O_final(σ) + {ji}
                S_final(σ) : = S_final(σ) \ {i}
            end
        end
    end
end
```

Stage 4 (tie-breaking):

```
xy : = minσ { ij | i,j ∈ {1,...,C}, i ≠ j }
for m : = 1 to C–1
begin
    for n : = m+1 to C
    begin
        if ( Pσ[m,n] ≈σ Pσ[n,m] ) then
        begin
            for i : = 1 to C
            begin
                for j : = 1 to C
                begin
                    if ( i ≠ j ) then
                    begin
                        forbidden[i,j] : = false
                        Qσ[i,j] : = Pσ[i,j]
                    end
                end
            end
            bool1 : = false
            while ( bool1 = false )
            begin
                for i : = 1 to C
                begin
                    for j : = 1 to C
                    begin
                        if ( i ≠ j ) then
                        begin
                            if ( Qσ[m,n] ≈σ ij ) then
                            begin
                                forbidden[i,j] : = true
                            end
                        end
                    end
                end
                for i : = 1 to C
                begin
                    for j : = 1 to C
                    begin
                        if ( i ≠ j ) then
                        begin
                            if ( forbidden[i,j] = true ) then
                            begin
                                Qσ[i,j] : = xy
                            end
                            else
                            begin
                                Qσ[i,j] : = ij
                            end
                        end
                    end
                end
```

```
100                 for i : = 1 to C
101                 begin
102                     for j : = 1 to C
103                     begin
104                         if ( i ≠ j ) then
105                         begin
106                             for k : = 1 to C
107                             begin
108                                 if ( i ≠ k ) then
109                                 begin
110                                     if ( j ≠ k ) then
111                                     begin
112                                         if ( Qσ[j,k] ≺σ minσ { Qσ[j,i], Qσ[i,k] } ) then
113                                         begin
114                                             Qσ[j,k] : = minσ { Qσ[j,i], Qσ[i,k] }
115                                         end
116                                     end
117                                 end
118                             end
119                         end
120                     end
121                 end
122                 if ( Qσ[m,n] ≻σ Qσ[n,m] ) then
123                 begin
124                     O_final(σ) : = O_final(σ) + {mn}
125                     S_final(σ) : = S_final(σ) \ {n}
126                     bool1 : = true
127                 end
128                 else
129                 if ( Qσ[m,n] ≺σ Qσ[n,m] ) then
130                 begin
131                     O_final(σ) : = O_final(σ) + {nm}
132                     S_final(σ) : = S_final(σ) \ {m}
133                     bool1 : = true
134                 end
135             end
136         end
137     end
138 end
```

For each pair of alternatives $m,n \in A$, we check whether $P_{\sigma}[m,n] \approx_{\sigma} P_{\sigma}[n,m]$ (lines 50–55). In this case, the link ij with $P_{\sigma}[m,n] \approx_{\sigma} ij$ is declared *forbidden* (lines 70–82) and the strongest paths, that don’t contain *forbidden* links, are calculated (lines 83–121). This procedure is repeated (lines 67–68) until this indifference is resolved (lines 122–134).

We define

(5.1.2) $\qquad \mathcal{O}_{final} := \cap \{ \mathcal{O}_{final}(\sigma) \mid \sigma \in \mathcal{LO}_{A \times A} \text{ with (5.1.1)} \}.$

(5.1.3) $\qquad \mathcal{S}_{final} := \cup \{ \mathcal{S}_{final}(\sigma) \mid \sigma \in \mathcal{LO}_{A \times A} \text{ with (5.1.1)} \}.$

5.2. Calculating a Tie-Breaking Ranking of the Candidates and a Tie-Breaking Ranking of the Links

The Schulze relation O, as defined in (2.2.1), is only a strict partial order. However, sometimes, a linear order is needed. In this section, we will show how the Schulze relation O can be completed to a linear order without having to sacrifice any of the desired criteria.

Step 1:

A *Tie-Breaking Ranking of the Links* (TBRL), a linear order $\succ_\sigma$ on $A \times A$, and a *Tie-Breaking Ranking of the Candidates* (TBRC), a linear order $\succ_\mu$ on A, are calculated as follows:

a) In the beginning:

- $\forall\ (i,j),(m,n) \in A \times A$: $(N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m]) \Rightarrow ij \succ_\sigma mn$.
- $\forall\ (i,j),(m,n) \in A \times A$: $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m]) \Rightarrow ij \approx_\sigma mn$.
- $\forall\ i,j \in A$: $i \approx_\mu j$.

b) Pick a random ballot $v \in V$ and use its rankings. That means:

- $\forall\ (i,j),(m,n) \in A \times A$: If $ij \approx_\sigma mn$ and

  (5.2.1) $\quad ((i \gtrsim_v j) \wedge (m \prec_v n)) \vee ((i \succ_v j) \wedge (m \lesssim_v n))$

  then replace “ $ij \approx_\sigma mn$ ” by “ $ij \succ_\sigma mn$ ”.

- $\forall\ i,j \in A$: If $i \approx_\mu j$ and $i \succ_v j$, then replace “ $i \approx_\mu j$ ” by “ $i \succ_\mu j$ ”.

When the bylaws require that the chairperson decides in the case of a tie, then, for the calculations of the TBRL and the TBRC, the ballot of the chairperson has to be chosen first.

c) Continue picking ballots randomly from those that have not yet been picked and use their rankings.

d) If you go through all ballots and there are still alternatives $i,j \in A$ with $i \approx_\mu j$, then proceed as follows:

d1) Pick a random alternative k and complete the TBRC in its favor. (That means: For all alternatives $l \in A \setminus \{k\}$ with $k \approx_\mu l$: Replace “ $k \approx_\mu l$ ” by “ $k \succ_\mu l$ ”.)

d2) Continue picking alternatives randomly from those that have not yet been picked and complete the TBRC in their favor.

Step 2:

Suppose there are still $(i,j),(m,n) \in A \times A$ with $ij \approx_{\sigma} mn$, then proceed as follows:

Variant 1: When at least one of the following conditions is satisfied, then replace “ $ij \approx_{\sigma} mn$ ” by “ $ij \succ_{\sigma} mn$ ”:

(5.2.2a) $i \succ_{\mu} j$ and $n \succ_{\mu} m$.
(5.2.3a) $i \succ_{\mu} j$ and $m \succ_{\mu} n$ and $i \succ_{\mu} m$.
(5.2.4a) $j \succ_{\mu} i$ and $n \succ_{\mu} m$ and $n \succ_{\mu} j$.
(5.2.5a) $i \equiv m$ and $n \succ_{\mu} j$.
(5.2.6a) $j \equiv n$ and $i \succ_{\mu} m$.

Variant 2: When at least one of the following conditions is satisfied, then replace “ $ij \approx_{\sigma} mn$ ” by “ $ij \succ_{\sigma} mn$ ”:

(5.2.2b) $i \succ_{\mu} j$ and $n \succ_{\mu} m$.
(5.2.3b) $i \succ_{\mu} j$ and $m \succ_{\mu} n$ and $n \succ_{\mu} j$.
(5.2.4b) $j \succ_{\mu} i$ and $n \succ_{\mu} m$ and $i \succ_{\mu} m$.
(5.2.5b) $i \equiv m$ and $n \succ_{\mu} j$.
(5.2.6b) $j \equiv n$ and $i \succ_{\mu} m$.

(5.2.2a) – (5.2.6a) and (5.2.2b) – (5.2.6b) are chosen in such a manner that e.g. when the TBRC $\succ_{\mu}$ is *abcdefgh* then links of otherwise equivalent strengths are sorted *ah*, *ag*, *af*, *ae*, *ad*, *ac*, *ab*, *bh*, *bg*, *bf*, *be*, *bd*, *bc*, *ch*, *cg*, *cf*, *ce*, *cd*, *dh*, *dg*, *df*, *de*, *eh*, *eg*, *ef*, *fh*, *fg*, *gh*, *hg*, *gf*, *hf*, *fe*, *ge*, *he*, *ed*, *fd*, *gd*, *hd*, *dc*, *ec*, *fc*, *gc*, *hc*, *cb*, *db*, *eb*, *fb*, *gb*, *hb*, *ba*, *ca*, *da*, *ea*, *fa*, *ga*, *ha* in variant 1 resp. *ah*, *bh*, *ch*, *dh*, *eh*, *fh*, *gh*, *ag*, *bg*, *cg*, *dg*, *eg*, *fg*, *af*, *bf*, *cf*, *df*, *ef*, *ae*, *be*, *ce*, *de*, *ad*, *bd*, *cd*, *ac*, *bc*, *ab*, *ba*, *cb*, *ca*, *dc*, *db*, *da*, *ed*, *ec*, *eb*, *ea*, *fe*, *fd*, *fc*, *fb*, *fa*, *gf*, *ge*, *gd*, *gc*, *gb*, *ga*, *hg*, *hf*, *he*, *hd*, *hc*, *hb*, *ha* in variant 2.

Step 3:

$\mathcal{O}_{final}(\sigma)$ and $\mathcal{S}_{final}(\sigma)$ are calculated as defined in section 5.1. The final winner is alternative $a \in A$ with $ba \notin \mathcal{O}_{final}(\sigma)$ for every $b \in A \setminus \{a\}$.

5.3. Transitivity

In section 4.1, we have proven that the binary relation $\mathcal{O}$, as defined in (2.2.1), is transitive. Nevertheless, it isn't intuitively clear whether also the binary relation $\mathcal{O}_{final}(\sigma)$, as defined in section 5.1, is transitive. It seems to be possible that ties $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$ are resolved based on different sets of *non-forbidden* links, so that the transitivity of $\mathcal{O}_{final}(\sigma)$ doesn't follow directly from the transitivity of $\mathcal{O}$.

However, in the following proof, we will see that also the binary relation $\mathcal{O}_{final}(\sigma)$, as defined in section 5.1, is transitive. We will prove that ties $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$ are either resolved based on the same set of *non-forbidden* links (sections 5.3.1, 5.3.4, and 5.3.5) or — in those cases, where these ties happen to be resolved based on different sets of *non-forbidden* links — they cannot violate transitivity (sections 5.3.2 and 5.3.3).

5.3.1. Part 1

Suppose, before we start declaring links *forbidden*, we have:

(5.3.1.1) $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

(5.3.1.2) $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

(5.3.1.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.1.1), we get $ab \in \mathcal{O}$ and, therefore, $ab \in \mathcal{O}_{final}(\sigma)$.

With (5.3.1.2), we get $bc \in \mathcal{O}$ and, therefore, $bc \in \mathcal{O}_{final}(\sigma)$.

This situation is not possible because, when no link has been declared *forbidden*, then all paths are calculated based on the same set of *non-forbidden* links. But in section 4.1, we have proven that, when all paths are calculated based on the same set of links, then the binary relation $\mathcal{O}$, as defined by $P_\sigma[x,y] \succ_\sigma P_\sigma[y,x]$, is transitive. So, with $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, we immediately get $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$.

5.3.2. Part 2

Suppose, before we start declaring links *forbidden*, we have:

(5.3.2.1) $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

(5.3.2.2) $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

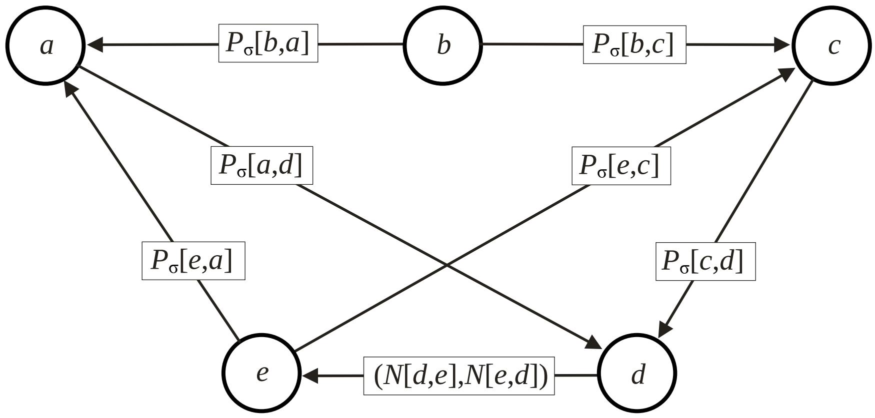
(5.3.2.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.2.1), we get $ba \in \mathcal{O}$ and, therefore, $ba \in \mathcal{O}_{final}(\sigma)$.

With (5.3.2.2), we get $bc \in \mathcal{O}$ and, therefore, $bc \in \mathcal{O}_{final}(\sigma)$.

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.2.1) – (5.3.2.3). Then the weakest link in the strongest path from alternative a to alternative c and the weakest link in the strongest path from alternative c to alternative a must be the same link, say de.

Therefore, the strongest paths have the following structure:



In this case, it can actually happen that the paths are based on different sets of *non-forbidden* links. In example 9 (section 3.9), we have a situation with $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ and where the link *de* is the weakest link in the strongest path from alternative *a* to alternative *c* and simultaneously the weakest link in the strongest path from alternative *c* to alternative *a*. So when we resolve $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* has to be declared *forbidden*. The strongest path from alternative *a* to alternative *c*, that doesn’t contain the link *de*, is *a*,$\underline{(24,21)}$,*c*. The strongest path from alternative *c* to alternative *a*, that doesn’t contain the link *de*, is *c*,(25,20),*b*,$\underline{(22,23)}$,*e*,(30,15),*a*. So $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $ac \in O_{final}(\sigma)$.

Now the interesting observation is that the link *de* is also in the strongest path from alternative *b* to alternative *a*. And the strongest path *b*,$\underline{(22,23)}$,*e*, (30,15),*a* from alternative *b* to alternative *a*, that doesn’t contain the link *de*, is weaker than the strongest path *a*,$\underline{(26,19)}$,*b* from alternative *a* to alternative *b*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a* based on the fact that the link *de* has been declared *forbidden* { what we don’t have to do, because each of (5.3.2.1) – (5.3.2.3) is resolved separately, based on its own set of *non-forbidden* links }, we would get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

Furthermore, the link *de* is in the strongest path from alternative *b* to alternative *c*. And the strongest path *b*,$\underline{(22,23)}$,*e*,(32,13),*c* from alternative *b* to alternative *c*, that doesn’t contain the link *de*, is weaker than the strongest path *c*,$\underline{(25,20)}$,*b* from alternative *c* to alternative *b*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *b* to alternative *c* and from alternative *c* to alternative *b* based on the fact that the link *de* has been declared *forbidden*, we would get $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

So example 9 (section 3.9) demonstrates that it can happen that (5.3.2.1) – (5.3.2.3) are resolved based on different sets of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ or to $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$ — transitivity will never be violated.

5.3.3. Part 3

Suppose, before we start declaring links *forbidden*, we have:

(5.3.3.1) $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$.

(5.3.3.2) $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

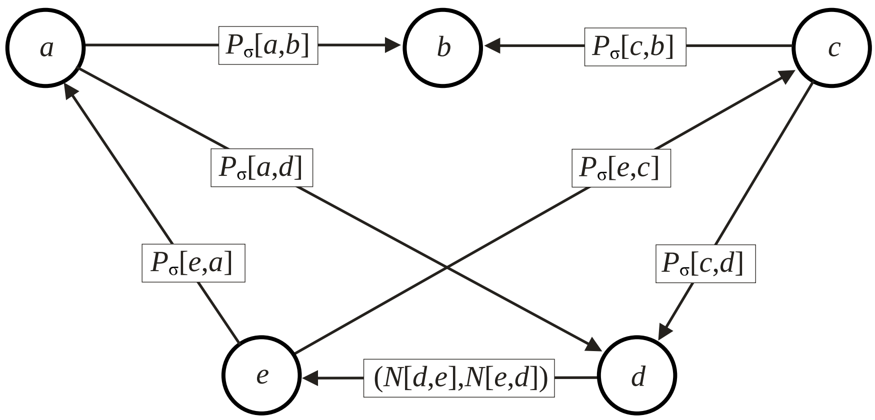
(5.3.3.3) $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

With (5.3.3.1), we get $ab \in O$ and, therefore, $ab \in O_{final}(\sigma)$.

With (5.3.3.2), we get $cb \in O$ and, therefore, $cb \in O_{final}(\sigma)$.

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.3.1) – (5.3.3.3). Then the weakest link in the strongest path from alternative *a* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *a* must be the same link, say *de*.

Therefore, the strongest paths have the following structure:



In this case, it can actually happen that the paths are based on different sets of *non-forbidden* links. In example 10 (section 3.10), we have a situation with $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ and where the link *de* is the weakest link in the strongest path from alternative *a* to alternative *c* and simultaneously the weakest link in the strongest path from alternative *c* to alternative *a*. So when we resolve $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* has to be declared *forbidden*. The strongest path from alternative *a* to alternative *c*, that doesn't contain the link *de*, is *a*,(24,21),*c*. The strongest path from alternative *c* to alternative *a*, that doesn't contain the link *de*, is *c*,(30,15),*d*,(22,23),*b*,(25,20),*a*. So $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $ac \in O_{final}(\sigma)$.

Now the interesting observation is that the link *de* is also in the strongest path from alternative *a* to alternative *b*. And the strongest path *a*,(32,13),*d*,(22,23),*b* from alternative *a* to alternative *b*, that doesn't contain the link *de*, is weaker than the strongest path *b*,(25,20),*a* from alternative *b* to alternative *a*, that doesn't contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *a* to alternative *b* and from alternative *b* to alternative *a* based on the fact that the link *de* has been declared *forbidden* { what we don't have to do, because each of (5.3.3.1) –

(5.3.3.3) is resolved separately, based on its own set of *non-forbidden* links }, we would get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

Furthermore, the link *de* is in the strongest path from alternative *c* to alternative *b*. And the strongest path *c*,(30,15),*d*,<u>(22,23)</u>,*b* from alternative *c* to alternative *b*, that doesn’t contain the link *de*, is weaker than the strongest path *b*,<u>(26,19)</u>,*c* from alternative *b* to alternative *c*, that doesn’t contain the link *de*. Therefore, if we had to recalculate the strengths of the strongest paths from alternative *b* to alternative *c* and from alternative *c* to alternative *b* based on the fact that the link *de* has been declared *forbidden*, we would get $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$.

So example 10 (section 3.10) demonstrates that it can happen that (5.3.3.1) – (5.3.3.3) are resolved based on different sets of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved to $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ or to $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$ — transitivity will never be violated.

5.3.4. Part 4

Suppose, before we start declaring links *forbidden*, we have:

(5.3.4.1) $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$.

(5.3.4.2) $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$.

(5.3.4.3) $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$.

With (5.3.4.3), we get $ca \in \mathcal{O}$ and, therefore, $ca \in \mathcal{O}_{final}(\sigma)$.

As the tie (5.3.4.1) and the tie (5.3.4.2) are resolved separately, it seems to be possible that they are resolved based on different sets of *non-forbidden* links, so that the transitivity of $\mathcal{O}_{final}(\sigma)$ doesn’t follow directly from the transitivity of $\mathcal{O}$. It seems to be possible that the tie (5.3.4.1) is resolved to $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and that simultaneously — as other links are declared *forbidden* during the process of resolving the tie (5.3.4.2), so that the strengths of the strongest paths are determined based on different sets of *non-forbidden* links — the tie (5.3.4.2) is resolved to $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, so that the transitivity of $\mathcal{O}_{final}(\sigma)$ is violated. However, the following proof shows that transitivity will never be violated.

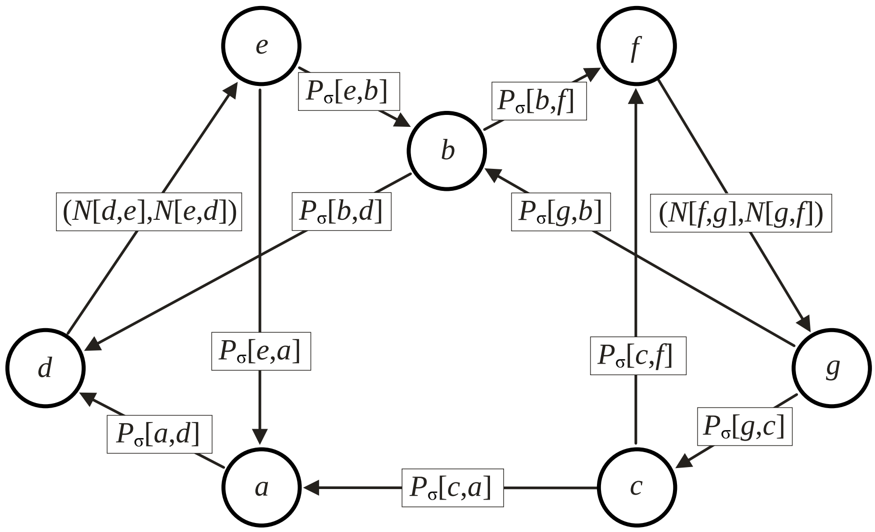
<u>**Claim:**</u>

Suppose (5.3.4.1) – (5.3.4.3) are resolved as prescribed in section 5.1. Then transitivity will never be violated.

<u>**Proof:**</u>

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.4.1) – (5.3.4.3). Then the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *de*. Furthermore, the weakest link in the strongest path from alternative *b* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *b* must be the same link, say *fg*.

Therefore, the strongest paths have the following structure:



As *de* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(5.3.4.4) $P_\sigma[a,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.4.5) $P_\sigma[e,b] \succ_\sigma (N[d,e],N[e,d])$.

As *de* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(5.3.4.6) $P_\sigma[b,d] \succ_\sigma (N[d,e],N[e,d])$.

(5.3.4.7) $P_\sigma[e,a] \succ_\sigma (N[d,e],N[e,d])$.

As *fg* is the weakest link in the strongest path from alternative *b* to alternative *c*, we get

(5.3.4.8) $P_\sigma[b,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.4.9) $P_\sigma[g,c] \succ_\sigma (N[f,g],N[g,f])$.

As *fg* is the weakest link in the strongest path from alternative *c* to alternative *b*, we get

(5.3.4.10) $P_\sigma[c,f] \succ_\sigma (N[f,g],N[g,f])$.

(5.3.4.11) $P_\sigma[g,b] \succ_\sigma (N[f,g],N[g,f])$.

With (5.3.4.4), (5.3.4.5), (5.3.4.8), and (5.3.4.9), we get: $a \to d \to e \to b \to f \to g \to c$ is a path from alternative *a* to alternative *c* with a strength of $\min_\sigma \{ (N[d,e],N[e,d]), (N[f,g],N[g,f]) \}$. Therefore, with (5.3.4.3), we get

(5.3.4.12) $P_\sigma[c,a] \succ_\sigma \min_\sigma \{ (N[d,e],N[e,d]), (N[f,g],N[g,f]) \}$.

Case 1: Suppose

(5.3.4.13a) $(N[d,e],N[e,d]) \succ_\sigma (N[f,g],N[g,f])$.

Then, with (5.3.4.12), (5.3.4.4), (5.3.4.13a), and (5.3.4.5), we get: $c \to a \to d \to e \to b$ is a path from alternative c to alternative b with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that fg is the weakest link in the strongest path from alternative c to alternative b.

Case 2: Suppose

(5.3.4.13b) $(N[d,e],N[e,d]) \prec_\sigma (N[f,g],N[g,f])$.

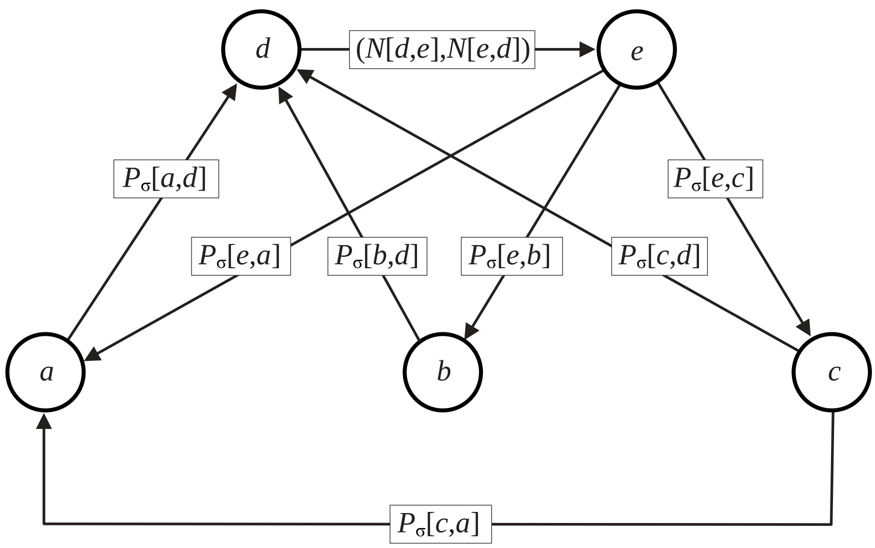
Then, with (5.3.4.8), (5.3.4.13b), (5.3.4.9), and (5.3.4.12), we get: $b \to f \to g \to c \to a$ is a path from alternative b to alternative a with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that de is the weakest link in the strongest path from alternative b to alternative a.

As (5.3.4.13a) and (5.3.4.13b) are not possible, we get

(5.3.4.13c) $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f])$.

As there are no links of equivalent strengths, (5.3.4.13c) means that de and fg are the same link. So to resolve (5.3.4.1) and (5.3.4.2), the same link is declared *forbidden*.

Therefore, the strongest paths have the following structure:



Without loss of generality, we can also say that the same link is declared *forbidden* in the process of resolving (5.3.4.3). The reason: With (5.3.4.12), we get that the link *de* cannot be in the strongest path from alternative *c* to alternative *a*. Therefore, the strongest path from alternative *c* to alternative *a* cannot be weakened by declaring the link *de forbidden*. The strongest path from alternative *a* to alternative *c* can be weakened by declaring the link *de forbidden*. But as we already know from (5.3.4.3) that the strongest path from alternative *c* to alternative *a* is stronger than the strongest path from alternative *a* to alternative *c*, declaring the link *de forbidden* cannot have an impact on the resolution of (5.3.4.3).

When the link *de* is declared *forbidden*, we get one of the following cases:

<u>Case A:</u> We still get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. In this case, with the same argumentation as in cases 1–2 we get that the same link, say *d’e’*, is the weakest link in the strongest path from alternative *a* to alternative *b*, the weakest link in the strongest path from alternative *b* to alternative *a*, the weakest link in the strongest path from alternative *b* to alternative *c*, and the weakest link in the strongest path from alternative *c* to alternative *b*. So we can proceed with declaring the link *d’e’ forbidden* until we get one of the cases B–G.

<u>Case B:</u> We get ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$). In this case, we succeeded in resolving (5.3.4.1) – (5.3.4.3) without violating transitivity.

<u>Case C:</u> We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So with $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ and $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$.

<u>Case D:</u> We get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So with $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$.

Case E: We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.4.1) – (5.3.4.3) are still calculated based on the same set of *non-forbidden* links. So $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$, $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$, and $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$ together violate transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links.

Case F: We get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.2. It is possible that $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn't matter whether $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ is resolved to $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ or to $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ — transitivity will never be violated.

Case G: We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.3. It is possible that $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn't matter whether $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$ is resolved to $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ or to $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$ — transitivity will never be violated.

The following table shows that cases A–G cover all possible combinations. Therefore, it has been proven for every possible situation that, when we resolve (5.3.4.1) – (5.3.4.3) as prescribed in section 5.1, then transitivity will never be violated.

$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case A
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case F
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case C
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case E
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case G
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case B

□

5.3.5. Part 5

Suppose, before we start declaring links *forbidden*, we have:

(5.3.5.1) $P_{\sigma}[a,b] \approx_{\sigma} P_{\sigma}[b,a]$.

(5.3.5.2) $P_{\sigma}[b,c] \approx_{\sigma} P_{\sigma}[c,b]$.

(5.3.5.3) $P_{\sigma}[c,a] \approx_{\sigma} P_{\sigma}[a,c]$.

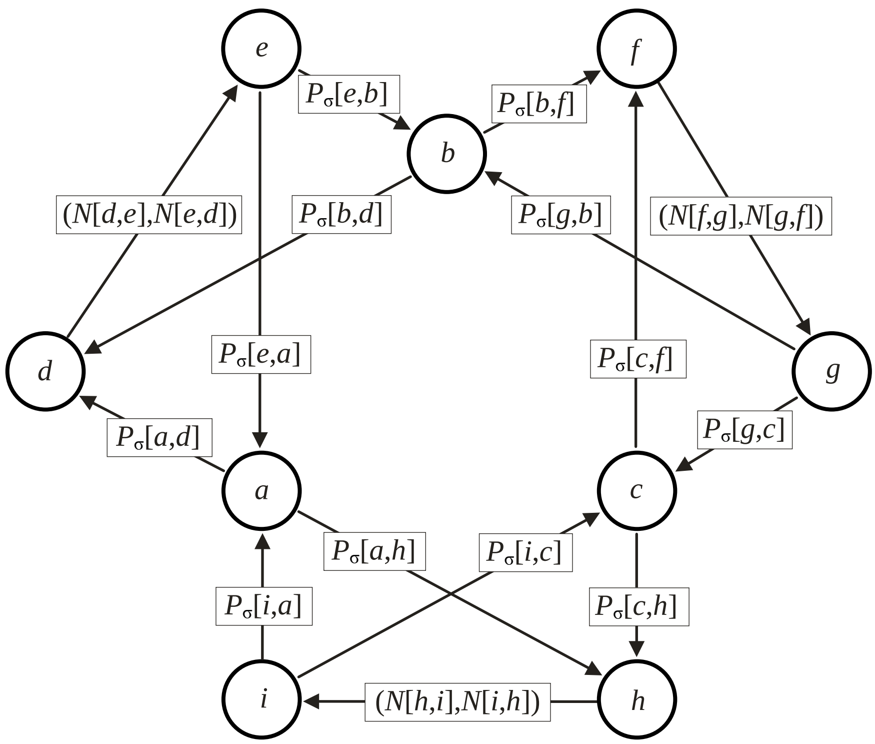
Claim:

Suppose (5.3.5.1) – (5.3.5.3) are resolved as prescribed in section 5.1. Then transitivity will never be violated.

Proof:

Suppose there are no pairwise links of equivalent strengths. Suppose (5.3.5.1) – (5.3.5.3). Then the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *de*. Furthermore, the weakest link in the strongest path from alternative *b* to alternative *c* and the weakest link in the strongest path from alternative *c* to alternative *b* must be the same link, say *fg*. Furthermore, the weakest link in the strongest path from alternative *c* to alternative *a* and the weakest link in the strongest path from alternative *a* to alternative *c* must be the same link, say *hi*.

Therefore, the strongest paths have the following structure:



As *de* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(5.3.5.4) $P_{\sigma}[a,d] \succ_{\sigma} (N[d,e],N[e,d])$.

(5.3.5.5) $P_{\sigma}[e,b] \succ_{\sigma} (N[d,e],N[e,d])$.

As *de* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(5.3.5.6) $P_{\sigma}[b,d] \succ_{\sigma} (N[d,e],N[e,d])$.

(5.3.5.7) $P_{\sigma}[e,a] \succ_{\sigma} (N[d,e],N[e,d])$.

As *fg* is the weakest link in the strongest path from alternative *b* to alternative *c*, we get

(5.3.5.8) $P_{\sigma}[b,f] \succ_{\sigma} (N[f,g],N[g,f])$.

(5.3.5.9) $P_{\sigma}[g,c] \succ_{\sigma} (N[f,g],N[g,f])$.

As *fg* is the weakest link in the strongest path from alternative *c* to alternative *b*, we get

(5.3.5.10) $P_{\sigma}[c,f] \succ_{\sigma} (N[f,g],N[g,f])$.

(5.3.5.11) $P_{\sigma}[g,b] \succ_{\sigma} (N[f,g],N[g,f])$.

As *hi* is the weakest link in the strongest path from alternative *c* to alternative *a*, we get

(5.3.5.12) $P_{\sigma}[c,h] \succ_{\sigma} (N[h,i],N[i,h])$.

(5.3.5.13) $P_{\sigma}[i,a] \succ_{\sigma} (N[h,i],N[i,h])$.

As *hi* is the weakest link in the strongest path from alternative *a* to alternative *c*, we get

(5.3.5.14) $P_{\sigma}[a,h] \succ_{\sigma} (N[h,i],N[i,h])$.

(5.3.5.15) $P_{\sigma}[i,c] \succ_{\sigma} (N[h,i],N[i,h])$.

Case 1: Suppose

(5.3.5.16a) $(N[d,e],N[e,d]) \prec_\sigma (N[f,g],N[g,f])$.

(5.3.5.17a) $(N[d,e],N[e,d]) \prec_\sigma (N[h,i],N[i,h])$.

Then, with (5.3.5.14), (5.3.5.17a), (5.3.5.15), (5.3.5.10), (5.3.5.16a), and (5.3.5.11), we get: $a \rightarrow h \rightarrow i \rightarrow c \rightarrow f \rightarrow g \rightarrow b$ is a path from alternative a to alternative b with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that de is the weakest link in the strongest path from alternative a to alternative b.

Similarly, with (5.3.5.8), (5.3.5.16a), (5.3.5.9), (5.3.5.12), (5.3.5.17a), and (5.3.5.13), we get: $b \rightarrow f \rightarrow g \rightarrow c \rightarrow h \rightarrow i \rightarrow a$ is a path from alternative b to alternative a with a strength of more than $(N[d,e],N[e,d])$. But this is a contradiction to the presumption that de is the weakest link in the strongest path from alternative b to alternative a.

Case 2: Suppose

(5.3.5.16b) $(N[f,g],N[g,f]) \prec_\sigma (N[d,e],N[e,d])$.

(5.3.5.17b) $(N[f,g],N[g,f]) \prec_\sigma (N[h,i],N[i,h])$.

Then, with (5.3.5.6), (5.3.5.16b), (5.3.5.7), (5.3.5.14), (5.3.5.17b), and (5.3.5.15), we get: $b \rightarrow d \rightarrow e \rightarrow a \rightarrow h \rightarrow i \rightarrow c$ is a path from alternative b to alternative c with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that fg is the weakest link in the strongest path from alternative b to alternative c.

Similarly, with (5.3.5.12), (5.3.5.17b), (5.3.5.13), (5.3.5.4), (5.3.5.16b), and (5.3.5.5), we get: $c \rightarrow h \rightarrow i \rightarrow a \rightarrow d \rightarrow e \rightarrow b$ is a path from alternative c to alternative b with a strength of more than $(N[f,g],N[g,f])$. But this is a contradiction to the presumption that fg is the weakest link in the strongest path from alternative c to alternative b.

Case 3: Suppose

(5.3.5.16c) $(N[h,i],N[i,h]) \prec_\sigma (N[d,e],N[e,d])$.

(5.3.5.17c) $(N[h,i],N[i,h]) \prec_\sigma (N[f,g],N[g,f])$.

Then, with (5.3.5.10), (5.3.5.17c), (5.3.5.11), (5.3.5.6), (5.3.5.16c), and (5.3.5.7), we get: $c \rightarrow f \rightarrow g \rightarrow b \rightarrow d \rightarrow e \rightarrow a$ is a path from alternative c to alternative a with a strength of more than $(N[h,i],N[i,h])$. But this is a contradiction to the presumption that hi is the weakest link in the strongest path from alternative c to alternative a.

Similarly, with (5.3.5.4), (5.3.5.16c), (5.3.5.5), (5.3.5.8), (5.3.5.17c), and (5.3.5.9), we get: $a \rightarrow d \rightarrow e \rightarrow b \rightarrow f \rightarrow g \rightarrow c$ is a path from alternative a to alternative c with a strength of more than $(N[h,i],N[i,h])$. But this is a contradiction to the presumption that hi is the weakest link in the strongest path from alternative a to alternative c.

With cases 1–3, we get that none of the links *de*, *fg*, *hi* can be weaker than the other two links. Without loss of generality, we can presume that the link *hi* is the strongest one of the links *de*, *fg*, *hi*. So we get

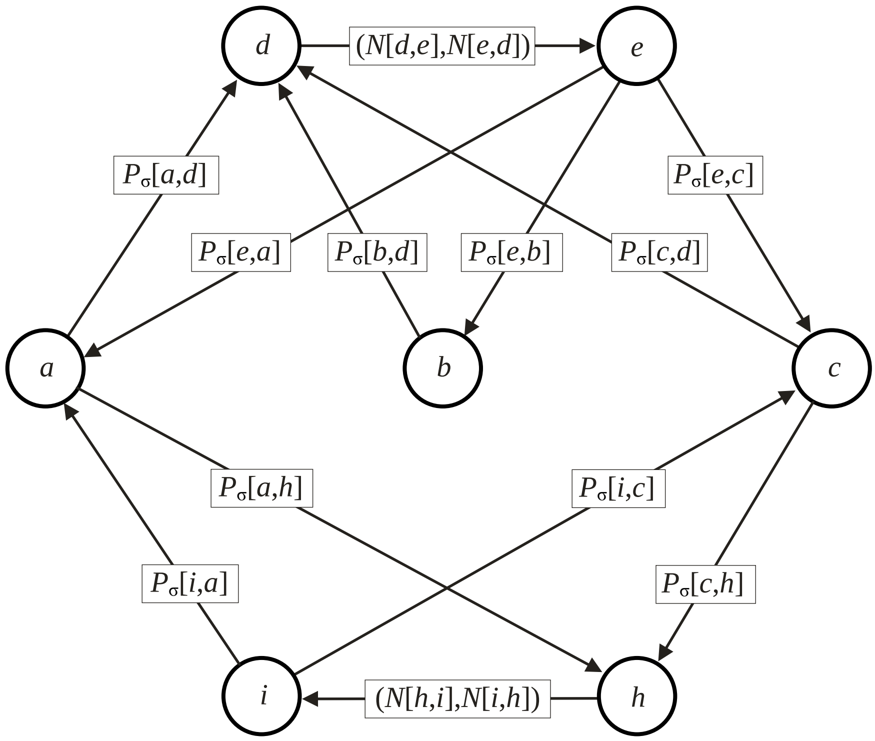
(5.3.5.18) $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \precsim_\sigma (N[h,i],N[i,h])$.

We can ignore the case $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \approx_\sigma (N[h,i],N[i,h])$ because in this case the links *de*, *fg*, *hi* are the same link so that for each of (5.3.5.1) – (5.3.5.3) the same link is declared *forbidden* first so that, afterwards, each of (5.3.5.1) – (5.3.5.3) is still resolved based on the same set of *non-forbidden* links.

So without loss of generality, we get

(5.3.5.19) $(N[d,e],N[e,d]) \approx_\sigma (N[f,g],N[g,f]) \prec_\sigma (N[h,i],N[i,h])$.

As there are no links of equivalent strengths, (5.3.5.19) means that the link *de* and the link *fg* must be the same link. Therefore, the strongest paths have the following structure:



Without loss of generality, we can also say that, when we resolve (5.3.5.1) – (5.3.5.3), then, at each stage, the weakest of the weakest links of the current strongest paths is declared *forbidden*. So in our situation, the link *de* is declared *forbidden* next.

Since $(N[d,e],N[e,d]) \prec_\sigma (N[h,i],N[i,h]) \approx_\sigma P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$, the link *de* cannot be in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*. Therefore, declaring the link *de forbidden* cannot have an impact on the strongest path from alternative *c* to alternative *a* or on the strongest path from alternative *a* to alternative *c*.

When the link *de* is declared *forbidden*, we get one of the following cases:

Case A: We still get $P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$. In this case, with the same argumentation as in cases 1–2 we get that the same link, say *d'e'*, is the weakest link in the strongest path from alternative *a* to alternative *b*, the weakest link in the strongest path from alternative *b* to alternative *a*, the weakest link in the strongest path from alternative *b* to alternative *c*, and the weakest link in the strongest path from alternative *c* to alternative *b*. So we can proceed with declaring the link *d'e' forbidden* until we get one of the cases B–F.

Case B: We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.5.1) – (5.3.5.3) are still calculated based on the same set of *non-forbidden* links. With $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[c,a] \prec_\sigma P_\sigma[a,c]$. But this is a contradiction to the fact that the link *de* cannot have been in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*, so that declaring the link *de forbidden* cannot have an impact on $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

Case C: We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is not possible because, after the link *de* has been declared *forbidden*, (5.3.5.1) – (5.3.5.3) are still calculated based on the same set of *non-forbidden* links. With $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$ and the transitivity, as proven in section 4.1 for cases where all paths are based on the same set of *non-forbidden* links, we would immediately get $P_\sigma[c,a] \succ_\sigma P_\sigma[a,c]$. But this is a contradiction to the fact that the link *de* cannot have been in the strongest path from alternative *c* to alternative *a* or in the strongest path from alternative *a* to alternative *c*, so that declaring the link *de forbidden* cannot have an impact on $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$.

<u>Case D:</u> We get ($P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$) or ($P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$). This case is not possible because we have seen in (5.3.4.13a) – (5.3.4.13c) that, when we have a situation with $P_\sigma[x,y] \approx_\sigma P_\sigma[y,x]$, $P_\sigma[y,z] \approx_\sigma P_\sigma[z,y]$, and $P_\sigma[z,x] \succ_\sigma P_\sigma[x,z]$, then the weakest link in the strongest path from alternative *x* to alternative *y*, the weakest link in the strongest path from alternative *y* to alternative *x*, the weakest link in the strongest path from alternative *y* to alternative *z*, and the weakest link in the strongest path from alternative *z* to alternative *y* must be the same link. But this is not possible because (5.3.5.19) says that the link *hi* is stronger than the link *de*.

<u>Case E:</u> We get $P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.2. It is possible that $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[a,c] \approx_\sigma P_\sigma[c,a]$ is resolved to $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$ or to $P_\sigma[a,c] \prec_\sigma P_\sigma[c,a]$ — transitivity will never be violated.

<u>Case F:</u> We get $P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$. This case is identical to the situation in section 5.3.3. It is possible that $P_\sigma[c,a] \approx_\sigma P_\sigma[a,c]$ is resolved based on a different set of *non-forbidden* links. However, this is not a problem because — it doesn’t matter whether $P_\sigma[a,c] \approx_\sigma P_\sigma[c,a]$ is resolved to $P_\sigma[a,c] \succ_\sigma P_\sigma[c,a]$ or to $P_\sigma[a,c] \prec_\sigma P_\sigma[c,a]$ — transitivity will never be violated.

The following table shows that cases A–F cover all possible combinations. Therefore, it has been proven for every possible situation that, when we resolve (5.3.5.1) – (5.3.5.3) as prescribed in section 5.1, then transitivity will never be violated.

$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case A
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \approx_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case B
$P_\sigma[a,b] \succ_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case F
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \approx_\sigma P_\sigma[c,b]$	→ case D
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \succ_\sigma P_\sigma[c,b]$	→ case E
$P_\sigma[a,b] \prec_\sigma P_\sigma[b,a]$ and $P_\sigma[b,c] \prec_\sigma P_\sigma[c,b]$	→ case C

□

5.4. Analysis

5.4.1. The Probabilistic Framework

An election method is simply a mapping from some input to some output. In section 2.1, we presumed that the output is (1) a strict partial order $\mathcal{O}$ on A and (2) a set $\emptyset \neq \mathcal{S} \subseteq A$ of potential winners. In the probabilistic framework, the output of an election method is a probability distribution $p[\mathcal{O}] \in \mathbb{R}$ on $\mathcal{LO}_A$, where $\mathcal{LO}_A$ is the set of linear orders on A.

We get

(5.4.1.1) $\forall\ \mathcal{O} \in \mathcal{LO}_A$: $p[\mathcal{O}] \geq 0$.

(5.4.1.2) $\sum$ ($p[\mathcal{O}]$ | $\mathcal{O} \in \mathcal{LO}_A$) = 1.

Suppose $q[a,b] \in \mathbb{R}$ is the probability for $ab \in \mathcal{O}$ (i.e. the probability that alternative $a \in A$ is ranked ahead of alternative $b \in A \setminus \{a\}$ in the collective ranking $\mathcal{O}$).

Then, we get

(5.4.1.3) $q[a,b] := \sum$ ($p[\mathcal{O}]$ | $\mathcal{O} \in \mathcal{LO}_A$ with $ab \in \mathcal{O}$).

(5.4.1.4) $\forall\ a,b \in A$: $q[a,b] \geq 0$.

(5.4.1.5) $\forall\ a,b \in A$: $q[a,b] + q[b,a] = 1$.

Suppose $r[a] \in \mathbb{R}$ is the probability that alternative $a \in A$ is elected.

Then, we get

(5.4.1.6) $r[a] := \sum$ ($p[\mathcal{O}]$ | $\mathcal{O} \in \mathcal{LO}_A$ with $ab \in \mathcal{O}$ for all $b \in A \setminus \{a\}$).

(5.4.1.7) $\forall\ a \in A$: $r[a] \geq 0$.

(5.4.1.8) $\sum$ ($r[a]$ | $a \in A$) = 1.

5.4.2. Resolvability

Definition:

An election method satisfies the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique linear order (i.e. without a linear order $\mathcal{O} \in \mathcal{LO}_A$ with $p[\mathcal{O}] = 1$) tends to zero as the number of voters in the profile tends to infinity.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies the resolvability criterion.

Proof (overview):

1. Suppose the number of alternatives is fixed. We prove that, when the number of voters in the profile tends to infinity, the probability, that there are links with equivalent strengths, goes to zero. So the probability, that there are links *ef* and *gh* with $ef \approx_\sigma gh$, goes to zero.

2. We prove that (1) the link *ij* cannot be in the strongest path from alternative *j* to alternative *i* and (2) the link *ji* cannot be in the strongest path from alternative *i* to alternative *j*. Therefore, when we resolve the tie $P_\sigma[i,j] \approx_\sigma P_\sigma[j,i]$, it can neither happen that the link *ij* is declared *forbidden* nor that the link *ji* is declared *forbidden*. Therefore, in worst case, when there are no other paths of *non-forbidden* links anymore, $P_\sigma[i,j] \approx_\sigma P_\sigma[j,i]$ is resolved to $ij \in \mathcal{O}$ when $ij \succ_\sigma ji$ and to $ji \in \mathcal{O}$ when $ij \prec_\sigma ji$. So the algorithm in section 5.1 always terminates before all links have been declared *forbidden*.

Remark:

When there is a unique linear order (i.e. a linear order $\mathcal{O} \in \mathcal{LO}_A$ with $p[\mathcal{O}] = 1$) then, with (5.4.1.6), we get that there is also a unique winner (i.e. an alternative $a \in A$ with $r[a] = 1$):

$$(\exists\, \mathcal{O} \in \mathcal{LO}_A\colon p[\mathcal{O}] = 1) \Rightarrow (\exists\, a \in A\colon r[a] = 1).$$

5.4.3. Pareto

In the probabilistic framework, the *Pareto criterion* says that, when no voter strictly prefers alternative $b \in A$ to alternative $a \in A$ [see (5.4.3.1)] and at least one voter strictly prefers alternative a to alternative b [see (5.4.3.2)], then $r[b] = 0$.

Definition:

An election method satisfies the *Pareto criterion* if the following holds:

Suppose:

(5.4.3.1) $\forall\ v \in V$: $a \succsim_v b$.

(5.4.3.2) $\exists\ v \in V$: $a \succ_v b$.

Then:

(5.4.3.3) $q[a,b] = 1$.

(5.4.3.4) $r[b] = 0$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies the Pareto criterion.

Proof (overview):

We prove

(5.4.3.5) $a \succ_\mu b$ with certainty.

With (4.3.2.8), (5.2.1), (5.2.6a), and (5.2.6b), we prove

(5.4.3.6) $\forall\ e \in A \setminus \{a,b\}$: $ae \succ_\sigma be$ with certainty.

With (4.3.2.9), (5.2.1), (5.2.5a), and (5.2.5b), we prove

(5.4.3.7) $\forall\ e \in A \setminus \{a,b\}$: $eb \succ_\sigma ea$ with certainty.

With (2.1.1), (5.2.1), (5.4.3.1), and (5.4.3.2), we prove

(5.4.3.8) $ab \succ_\sigma ba$ with certainty.

With (5.4.3.6), (5.4.3.7), and (5.4.3.8), we prove

(5.4.3.9) $ab \in \mathcal{O}$ with certainty.

5.4.4. Reversal Symmetry

In the probabilistic framework, *reversal symmetry* says that, when $\succ_v$ is reversed for all $v \in V$, then $r^{old}[a] + r^{new}[a] \leq 1$ for all $a \in A$. Otherwise, if $r^{old}[a] + r^{new}[a]$ was larger than 1 for some alternative $a \in A$, then this would mean that, with a probability of at least $r^{old}[a] + r^{new}[a] - 1 > 0$, alternative a is identified as best alternative and, simultaneously, identified as worst alternative.

Suppose $\mathcal{O}^{reverse} \in \mathcal{LO}_A$ is the reversal of $\mathcal{O} \in \mathcal{LO}_A$.

That means:

(5.4.4.1) $\forall\, a,b \in A$: $ab \in \mathcal{O} \Leftrightarrow ba \in \mathcal{O}^{reverse}$.

Definition:

An election method satisfies *reversal symmetry* if the following holds:

Suppose:

(5.4.4.2) $\forall\, e,f \in A\ \forall\, v \in V$: $e \succ_v^{old} f \Leftrightarrow f \succ_v^{new} e$.

Then:

(5.4.4.3) $\forall\, \mathcal{O} \in \mathcal{LO}_A$: $p^{old}[\mathcal{O}] = p^{new}[\mathcal{O}^{reverse}]$.

(5.4.4.4) $\forall\, a,b \in A$: $q^{old}[a,b] = q^{new}[b,a]$.

(5.4.4.5) $\forall\, a \in A$: $r^{old}[a] + r^{new}[a] \leq 1$.

Claim:

Suppose $\succ_D$ satisfies (2.1.2). Suppose, for every $(i,j),(m,n) \in A \times A$, there is at least one voter $v \in V$ with

(5.4.4.6)
$$\begin{aligned}
&(\,(\,i \succ_v j\,) \wedge (\,m \prec_v n\,)\,)\\
&\vee (\,(\,i \succ_v j\,) \wedge (\,m \approx_v n\,)\,)\\
&\vee (\,(\,i \approx_v j\,) \wedge (\,m \prec_v n\,)\,)\\
&\vee (\,(\,i \prec_v j\,) \wedge (\,m \succ_v n\,)\,)\\
&\vee (\,(\,i \prec_v j\,) \wedge (\,m \approx_v n\,)\,)\\
&\vee (\,(\,i \approx_v j\,) \wedge (\,m \succ_v n\,)\,).
\end{aligned}$$

Then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies reversal symmetry.

Proof (overview):

Suppose, for every $(i,j),(m,n) \in A \times A$, there is at least one voter $v \in V$ with (5.4.4.6). Then it is guaranteed that (5.2.1) resolves every $(i,j),(m,n) \in A \times A$ to $ij \succ_\sigma mn$ or $ij \prec_\sigma mn$. So the TBRL $\succ_\sigma$, as determined in step 1 of section 5.2, is already linear.

Furthermore, (2.1.2) guarantees that, when $\succ_v$ is reversed for all $v \in V$, also the TBRL $\succ_\sigma$, as determined in step 1 of section 5.2, is reversed.

So the probability that O is chosen in the original situation is identical to the probability that $O^{reverse}$ is chosen in the reversed situation. As we have presumed in section 2.1 that there are at least 2 alternatives in A, $a \in A$ cannot be the maximum element of O and simultaneously the maximum element of $O^{reverse}$. Therefore, we get (5.4.4.5).

Example 12:

(α) When we apply the proposed method to example 12 (section 3.12), we first calculate the TBRL $\succ_\sigma$.

We have:

$(N[b,c],N[c,b]) \approx_D (4,1)$.
$(N[a,b],N[b,a]) \approx_D (3,2)$.
$(N[c,a],N[a,c]) \approx_D (3,2)$.
$(N[a,c],N[c,a]) \approx_D (2,3)$.
$(N[b,a],N[a,b]) \approx_D (2,3)$.
$(N[c,b],N[b,c]) \approx_D (1,4)$.

So we start with $bc \succ_\sigma ab \approx_\sigma ca \succ_\sigma ac \approx_\sigma ba \succ_\sigma cb$.

Case I: With a probability of 2/5, one of the $a \succ_v b \succ_v c$ voters is chosen first. $ab \approx_\sigma ca$ is then completed to $ab \succ_\sigma ca$ because this voter supports the link ab and opposes the link ca. $ac \approx_\sigma ba$ is completed to $ac \succ_\sigma ba$ because this voter supports the link ac and opposes the link ba. So the TBRL $\succ_\sigma$ is completed to $bc \succ_\sigma ab \succ_\sigma ca \succ_\sigma ac \succ_\sigma ba \succ_\sigma cb$.

Case II: With a probability of 2/5, one of the $b \succ_v c \succ_v a$ voters is chosen first. $ab \approx_\sigma ca$ is then completed to $ca \succ_\sigma ab$ because this voter supports the link ca and opposes the link ab. $ac \approx_\sigma ba$ is completed to $ba \succ_\sigma ac$ because this voter supports the link ba and opposes the link ac. So the TBRL $\succ_\sigma$ is completed to $bc \succ_\sigma ca \succ_\sigma ab \succ_\sigma ba \succ_\sigma ac \succ_\sigma cb$.

Case III: With a probability of 1/5, the $c \succ_v a \succ_v b$ voter is chosen first. As this voter supports both links ab and ca, this voter cannot be used to complete $ab \approx_\sigma ca$. As this voter opposes both links ac and ba, this voter cannot be used to complete $ac \approx_\sigma ba$. With a probability of 1/2, one of the $a \succ_v b \succ_v c$ voters is chosen second; the TBRL $\succ_\sigma$ is then completed to $bc \succ_\sigma ab \succ_\sigma ca \succ_\sigma ac \succ_\sigma ba \succ_\sigma cb$ as described in Case I. With a probability of 1/2, one of the $b \succ_v c \succ_v a$ voters is chosen second; the TBRL $\succ_\sigma$ is then completed to $bc \succ_\sigma ca \succ_\sigma ab \succ_\sigma ba \succ_\sigma ac \succ_\sigma cb$ as described in Case II.

So with a probability of 1/2, the TBRL $\succ_\sigma$ is completed to $bc \succ_\sigma ab \succ_\sigma ca \succ_\sigma ac \succ_\sigma ba \succ_\sigma cb$ and, with a probability of 1/2, the TBRL $\succ_\sigma$ is completed to $bc \succ_\sigma ca \succ_\sigma ab \succ_\sigma ba \succ_\sigma ac \succ_\sigma cb$.

(β) When the TBRL $bc \succ_\sigma ab \succ_\sigma ca \succ_\sigma ac \succ_\sigma ba \succ_\sigma cb$ is used. The weakest link in the strongest path from alternative *a* to alternative *b* is *ab*. The weakest link in the strongest path from alternative *b* to alternative *a* is *ca*. As $ab \succ_\sigma ca$, we get $P_D[a,b] \succ_D P_D[b,a]$. Alternative *a* is the final winner.

When the TBRL $bc \succ_\sigma ca \succ_\sigma ab \succ_\sigma ba \succ_\sigma ac \succ_\sigma cb$ is used. The weakest link in the strongest path from alternative *a* to alternative *b* is *ab*. The weakest link in the strongest path from alternative *b* to alternative *a* is *ca*. As $ca \succ_\sigma ab$, we get $P_D[b,a] \succ_D P_D[a,b]$. Alternative *b* is the final winner.

So in example 12, we get: $r^{\text{old}}[a] = 0.5$ and $r^{\text{old}}[b] = 0.5$.

(γ) When the individual ballots are reversed, we get:

Example 12 (new):

2	voters	$c \succ_v b \succ_v a$
2	voters	$a \succ_v c \succ_v b$
1	voter	$b \succ_v a \succ_v c$

When we rename the alternatives *b* and *c* and reorder the voters, we see that example 12 (new) is identical to example 12. So with anonymity and neutrality, we get $r^{\text{new}}[a] = r^{\text{old}}[a]$, $r^{\text{new}}[b] = r^{\text{old}}[c]$, and $r^{\text{new}}[c] = r^{\text{old}}[b]$. So we get: $r^{\text{new}}[a] = 0.5$ and $r^{\text{new}}[c] = 0.5$.

(δ) The interesting conclusion is that anonymity, neutrality, and reversal symmetry together imply $r^{\text{old}}[a] \leq 0.5$ in example 12, because anonymity and neutrality together imply $r^{\text{new}}[a] = r^{\text{old}}[a]$ and reversal symmetry implies $r^{\text{old}}[a] + r^{\text{new}}[a] \leq 1$.

5.4.5. Monotonicity

In the probabilistic framework, *monotonicity* says that, when some voters rank alternative $a \in A$ higher [see (4.5.1) and (4.5.2)] without changing the order in which they rank the other alternatives relatively to each other [see (4.5.3)], then $r[a]$ must not decrease.

Definition:

An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified as described in (4.5.1) – (4.5.3). Then

(5.4.5.1) $\forall \emptyset \neq B \subseteq A \setminus \{a\}$:

$\sum (p^{old}[O] \mid O \in \mathcal{L}O_A \text{ with } ab \in O \text{ for all } b \in B)$

$\leq \sum (p^{new}[O] \mid O \in \mathcal{L}O_A \text{ with } ab \in O \text{ for all } b \in B)$.

(5.4.5.2) $\forall b \in A \setminus \{a\}$: $q^{old}[a,b] \leq q^{new}[a,b]$.

(5.4.5.3) $r^{old}[a] \leq r^{new}[a]$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method $O_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, satisfies monotonicity.

Proof (overview):

We prove, that when the ballots are modified as described in (4.5.1) – (4.5.3), then links af with $f \in A \setminus \{a\}$ can only rise in the TBRL $\succ_\sigma$ compared to other links eg with $e \in A \setminus \{a\}$ and $g \in A \setminus \{e\}$. Links fa with $f \in A \setminus \{a\}$ can only fall in the TBRL $\succ_\sigma$ compared to other links eg with $g \in A \setminus \{a\}$ and $e \in A \setminus \{g\}$. Links eg with $e \in A \setminus \{a\}$ and $g \in A \setminus \{a,e\}$ neither rise nor fall in the TBRL $\succ_\sigma$ compared to other links ij with $i \in A \setminus \{a\}$ and $j \in A \setminus \{a,i\}$.

The rest of the proof is identical to the proof in section 4.5.

5.4.6. Independence of Clones

Definition:

An election method is *independent of clones* if the following holds:

Suppose $d \in A^{old}$. Suppose $A^{new} := (A^{old} \cup K) \setminus \{d\}$.

Suppose alternative d is replaced by the set of alternatives K in such a manner that (4.6.1) – (4.6.3) are satisfied.

Then:

(5.4.6.1) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{(A^{old} \setminus \{d\})}\ \forall\, B \subseteq A^{old} \setminus \{d\}\ \forall\, g \in K$:

$$p^{old}[\mathcal{O}] \text{ for } \mathcal{O} \in \mathcal{LO}_{A^{old}} \text{ with } \begin{array}{l} (1)\ \mathcal{O}_1 \subset \mathcal{O} \text{ and} \\ (2)\ ad \in \mathcal{O} \text{ for all } a \in B \text{ and} \\ (3)\ db \in \mathcal{O} \text{ for all } b \notin B \end{array}$$

$$= \sum \left(p^{new}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{A^{new}} \text{ with } \begin{array}{l} (1)\ \mathcal{O}_1 \subset \mathcal{O} \text{ and} \\ (2)\ ag \in \mathcal{O} \text{ for all } a \in B \text{ and} \\ (3)\ gb \in \mathcal{O} \text{ for all } b \notin B \end{array} \right).$$

(5.4.6.2) $\forall\, a,b \in A^{old} \setminus \{d\}$: $q^{old}[a,b] = q^{new}[a,b]$.

(5.4.6.3) $\forall\, a \in A^{old} \setminus \{d\}\ \forall\, g \in K$: $q^{old}[a,d] = q^{new}[a,g]$.

(5.4.6.4) $\forall\, a \in A^{old} \setminus \{d\}$:
$(((r^{old}[a] = 0) \vee (\exists\, v \in V: a \not\approx_v^{old} d)) \Rightarrow (r^{old}[a] = r^{new}[a]))$.

Remark:

The presumption $((r^{old}[a] = 0) \vee (\exists\, v \in V: a \not\approx_v^{old} d))$ is needed to exclude situations where alternative a is chosen with positive probability (i.e.: $r^{old}[a] > 0$) and every voter is indifferent between alternative a and alternative d (i.e.: $a \approx_v^{old} d$ for every $v \in V$). In those situations, alternative a and alternative d are necessarily chosen with the same probability (i.e.: $r^{old}[a] = r^{old}[d]$). When alternative d is replaced by a set K of more than one alternative in such a manner that (4.6.1) – (4.6.3) are satisfied then, again, every alternative in $(K \cup \{a\})$ is necessarily chosen with the same probability (i.e.: $r^{new}[a] = r^{new}[g]$ for every $g \in K$), so that the probability, that alternative a is chosen, necessarily drops (i.e.: $r^{old}[a] > r^{new}[a]$).

Claim:

The Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_\sigma$, as defined in section 5.2, is independent of clones.

Proof (overview):

We prove that all the alternatives $g \in K$ are ranked in a consecutive manner in the TBRC $\succ_{\mu}$. We then prove that, for every $a \in A^{old} \setminus \{d\}$, all the links ag with $g \in K$ are ranked in a consecutive manner in the TBRL $\succ_{\sigma}$. We further prove that, for every $a \in A^{old} \setminus \{d\}$, all the links ga with $g \in K$ are ranked in a consecutive manner in the TBRL $\succ_{\sigma}$.

The rest of the proof is identical to the proof in section 4.6.

5.4.7. Smith

Definition:

An election method satisfies *Smith* if the following holds:

Suppose (4.7.1) and (4.7.2).

Then we get:

(5.4.7.1) $\forall\, a \in B_1\ \forall\, b \in B_2$: $q[a,b] = 1$.

(5.4.7.2) $\sum\,(\, r[a] \mid a \in B_1\,) = 1$.

An election method satisfies *Smith-IIA* if the following holds:

Suppose (4.7.1) and (4.7.2).

Suppose $d \in B_2$ is removed. Then we get:

(5.4.7.3) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{B_1}$:
$\sum\,(\, p^{old}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_A$ with $\mathcal{O}_1 \subset \mathcal{O}\,) =$
$\sum\,(\, p^{new}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{(A \setminus \{d\})}$ with $\mathcal{O}_1 \subset \mathcal{O}\,)$.

(5.4.7.4) $\forall\, a,b \in B_1$: $q^{old}[a,b] = q^{new}[a,b]$.

(5.4.7.5) $\forall\, a \in B_1$: $r^{old}[a] = r^{new}[a]$.

Suppose $d \in B_1$ is removed. Then we get:

(5.4.7.6) $\forall\, \mathcal{O}_1 \in \mathcal{LO}_{B_2}$:
$\sum\,(\, p^{old}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_A$ with $\mathcal{O}_1 \subset \mathcal{O}\,) =$
$\sum\,(\, p^{new}[\mathcal{O}] \mid \mathcal{O} \in \mathcal{LO}_{(A \setminus \{d\})}$ with $\mathcal{O}_1 \subset \mathcal{O}\,)$.

(5.4.7.7) $\forall\, a,b \in B_2$: $q^{old}[a,b] = q^{new}[a,b]$.

Claim:

If $\succ_D$ satisfies (2.1.5), then the Schulze method $\mathcal{O}_{final}(\sigma)$, as defined in sections 5.1, with the TBRL $\succ_{\sigma}$, as defined in section 5.2, satisfies Smith and Smith-IIA.

Proof (overview):

The proof is identical to the proofs in section 4.7.

5.4.8. Runtime

The runtime to calculate the pairwise matrix is O(N·(C^2)).

The runtime to calculate the TBRL is O(N·(C^4)) because, in worst case, O(N) ballots have to be picked and, each time, O(C^2) links are compared with O(C^2) other links, according to (5.2.1).

On closer examination, to sort the O(C^2) links according to their strengths, it is not necessary to compare each of the O(C^2) links with each other of the O(C^2) links directly. As the fastest algorithms to sort X items according to their strengths have a runtime of O(X·log(X)), the runtime of the fastest algorithms to sort the O(C^2) links according to their strengths is O((C^2)·log(C)).

Therefore, the runtime to calculate the TBRL, as defined in (5.2.1), reduces to O(N·(C^2)·log(C)).

The runtime to calculate a complete ranking, as defined in section 5.1, is O(C^7) because, in worst case, there are O(C^2) pairwise ties “$P_\sigma[m,n] \approx_\sigma P_\sigma[n,m]$” (line 54). In worst case, O(C^2) links have to be declared *forbidden* to resolve a pairwise tie. Each time, the runtime of the Floyd-Warshall algorithm to calculate the strength of the strongest path from every alternative to every other alternative is O(C^3).

On closer examination, to resolve the pairwise tie “$P_\sigma[m,n] \approx_\sigma P_\sigma[n,m]$”, it is not necessary to calculate the strength of the strongest path from every alternative to every other alternative. It is sufficient to calculate the strength of the strongest path from alternative m to alternative n and the strength of the strongest path from alternative n to alternative m. This can be done with the Dijkstra algorithm in a runtime O(C^2).

Therefore, the runtime to calculate a complete ranking, as defined in section 5.1, reduces to O(C^6).

Thus, the total runtime to calculate the binary relation O, as defined in section 5, is O((N·(C^2)·log(C)) + (C^6)).

6. Definition of the Strength of a Pairwise Link

6.1. Winning Votes

There has been some debate about how to define $\succ_D$ when it is presumed that on the one side each voter has a sincere linear order of the alternatives, but on the other side some voters cast only a strict weak order because of strategic considerations. We got to the conclusion that the strength ($N[e,f]$, $N[f,e]$) of the pairwise link $ef \in A \times A$ should be measured by *winning votes*, i.e. primarily by the support $N[e,f]$ of this link and secondarily by the opposition $N[f,e]$ to this link.

> $(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:
>
> 1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
> 2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
> 3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
> 4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.
> 5. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] < N[h,g]$.
> 6. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

Suppose $a,b \in A$. Suppose $R_1[a] := \| \{ v \in V \mid \forall c \in A \setminus \{a\}: a \succ_v c \} \|$ is the number of voters who strictly prefer alternative a to every other alternative. Suppose $R_2[b] := \| \{ v \in V \mid \exists c \in A \setminus \{b\}: b \succ_v c \} \|$ is the number of voters who strictly prefer alternative b to at least one other alternative. Suppose $R_1[a] > R_2[b]$. Then *Woodall’s plurality criterion* says: $b \notin \mathcal{S}$. Woodall (1997) writes: “If some candidate b has strictly fewer votes in total than some other candidate a has first-preference votes, then candidate b should not be elected.”

Claim:

If $\succ_{win}$ is being used, then the Schulze method satisfies Woodall’s plurality criterion.

Proof:

Suppose

(6.1.1) $R_1[a] > R_2[b]$.

With (6.1.1) and the definition for $\succ_{win}$, we get

(6.1.2) $(R_1[a],R_2[b]) \succ_{win} (R_2[b],0)$.

With the definitions for $R_1[a]$ and $R_2[b]$, we get

(6.1.3) $N[a,b] \geq R_1[a]$.

(6.1.4) $N[b,a] \leq R_2[b]$.

With (6.1.3), (6.1.4), and the definition for $\succ_{win}$, we get

(6.1.5) $(N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b])$.

With the definition for $R_2[b]$, we get

(6.1.6) $\forall c \in A \setminus \{b\}: N[b,c] \leq R_2[b]$.

With (6.1.6) and the definition for $\succ_{win}$, we get

(6.1.7) $\forall c \in A \setminus \{b\}: (N[b,c],N[c,b]) \precsim_{win} (R_2[b],0)$.

With (2.2.6) and (6.1.7), we get

(6.1.8) $P_{win}[b,a] \precsim_{win} (R_2[b],0)$.

With (2.2.3), (6.1.5), (6.1.2), and (6.1.8), we get

(6.1.9) $P_{win}[a,b] \succsim_{win} (N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b])$
$\succ_{win} (R_2[b],0) \succsim_{win} P_{win}[b,a]$

so that $ab \in \mathcal{O}$. □

6.2. Margins

Reversal independence says that adding a ballot and its reverse should not change the result of the elections. In other words, a ballot and its reverse should always cancel each other out.

Definition:

Suppose w_1 and w_2 are strict weak orders with

(6.2.1) $\forall\ a,b \in A$: $a \succ_{w_1} b \Leftrightarrow b \succ_{w_2} a$.

Suppose $V^{\text{new}} := V^{\text{old}} + \{w_1\} + \{w_2\}$.

Then, an election method satisfies *reversal independence* if the following holds:

(6.2.2) $\mathcal{O}^{\text{new}} = \mathcal{O}^{\text{old}}$.

(6.2.3) $\mathcal{S}^{\text{new}} = \mathcal{S}^{\text{old}}$.

Claim:

If $\succ_{margin}$ is being used, then the Schulze method, as defined in section 2.2, satisfies reversal independence.

Proof:

The proof is trivial. When w_1 and w_2 are added, then $N^{\text{new}}[a,b] - N^{\text{new}}[b,a] = N^{\text{old}}[a,b] - N^{\text{old}}[b,a]$ for all $a,b \in A$. Therefore

(6.2.4) $\forall\ (e,f),(g,h) \in A \times A$:

$$((N^{\text{new}}[e,f] - N^{\text{new}}[f,e] > N^{\text{new}}[g,h] - N^{\text{new}}[h,g])$$

$$\Leftrightarrow (N^{\text{old}}[e,f] - N^{\text{old}}[f,e] > N^{\text{old}}[g,h] - N^{\text{old}}[h,g])).$$

Therefore

(6.2.5) $\forall\ (e,f),(g,h) \in A \times A$:

$$(N^{\text{new}}[e,f],N^{\text{new}}[f,e]) \succ_{margin} (N^{\text{new}}[g,h],N^{\text{new}}[h,g])$$

$$\Leftrightarrow (N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \succ_{margin} (N^{\text{old}}[g,h],N^{\text{old}}[h,g]).$$

With (2.2.2) and (6.2.5), we get (6.2.2) and (6.2.3). □

7. Supermajority Requirements

When preferential ballots are being used in referendums, then sometimes alternatives have to fulfill some supermajority requirements to qualify. Typical supermajority requirements define some $M_1 \in \mathbb{N}$ or some $1 \leq M_2 \in \mathbb{R}$ and say that $N[a,b]$ must be strictly larger than max $\{ N[b,a], M_1 \}$ or that $N[a,b]$ must be strictly larger than $M_2 \cdot N[b,a]$ to replace alternative $b \in A$ by alternative $a \in A$. Or they say that $N[a,b]$ must be strictly larger than $N[b,a]$ not only in the electorate as a whole, but also in a majority of its geographic parts or even in each of its geographic parts. It is also possible that in the same referendum the voters have to choose between alternatives that have to fulfill different supermajority requirements to qualify. In this section, we discuss a possible way to combine the Schulze method with supermajority requirements. Suppose $s \in A$ is the *status quo*.

These are the two tasks of supermajority requirements:

Task #1 (*protecting the status quo*):

> Supermajority requirements protect the status quo from accidental majorities. They make it more difficult to replace the status quo s by alternative $a \in A \setminus \{s\}$. Therefore, an important property of all supermajority requirements is that, when s had won in the absence of these requirements, then it also wins in the presence of these requirements.

Task #2 (*preventing the status quo from cycling*):

> Supermajority requirements prevent the status quo from cycling. Suppose $s(0)$ is the starting status quo. Suppose $s(k+1)$ is the new status quo when the method is applied to the same set of alternatives A, to the same set of ballots V, and to the status quo $s(k)$. Then we would expect that (for every possible set of alternatives A, for every possible set of ballots V, and for every possible starting status quo $s(0) \in A$) there is an $m < C$ such that $s(k) \equiv s(m)$ for all $k \geq m$.

We recommend the following method:

> The Schulze relation $\mathcal{O}$, as defined in section 2.2, is calculated.
>
> A *Tie-Breaking Ranking of the Links* (TBRL), a linear order $\succ_\sigma$ on $A \times A$, and a *Tie-Breaking Ranking of the Candidates* (TBRC), a linear order $\succ_\mu$ on A, are calculated as described in section 5.2 variant 1.
>
> The final Schulze relation $\mathcal{O}_{final}(\sigma)$, as defined in section 5.1, is calculated.
>
> Alternative $a \in A \setminus \{s\}$ is *attainable* if and only if $N[a,s] > N[s,a]$ and (a) there is no supermajority requirement to replace the status quo s by alternative a or (b) alternative a has the supermajority required to replace the status quo s by alternative a.
>
> Alternative $a \in A$ is *eligible* if and only if ($a \equiv s$) or ((a is attainable) and ($as \in \mathcal{O}$)).
>
> A winner is an alternative $a \in A$ with (1) alternative a is eligible and (2) $ab \in \mathcal{O}_{final}(\sigma)$ for every other eligible alternative b.

The condition “$as \in \mathcal{O}$” in the definition of eligibility implies that alternative a can win only if it had disqualified the status quo s in the absence of supermajority requirements. This guarantees that, if s had won in the absence of supermajority requirements, then s also wins in the presence of these supermajority requirements.

In the above suggestion, the status quo s can only be replaced by an alternative a with $as \in \mathcal{O}$. As $\mathcal{O}$ is transitive, it is guaranteed that the status quo cannot be changed in a cyclic manner.

8. Electoral College

There has been some debate about how to combine the Schulze method with the Electoral College for the elections of the President of the USA. In my opinion, the Electoral College serves two important purposes:

> Purpose #1: The Electoral College gives more power to the smaller states.
>
>> The Senate, where each state has the same voting power regardless of its population, is more powerful than the House of Representatives, where each state has a voting power in proportion of its population. This is true especially for decisions that are close to the executive. For example, the President needs the consent of the Senate for treaties and for the appointment of officers and judges. Because of this reason, it is more important that the President has a reliable support in the Senate than that he has a reliable support in the House of Representatives.
>
> Purpose #2: The Electoral College makes it possible to count the ballots on the state levels and then to add up the electoral votes.
>
>> The Electoral College makes it possible that, to guarantee that all voters are treated in an equal manner, it is only necessary to guarantee that all voters *in the same state* are treated in an equal manner. However, if the ballots were added up on the national level, it would be necessary to guarantee that *all voters all over the USA* are treated in an equal manner. In the latter case, many provisions (e.g. the rules to gain suffrage or to be excluded from suffrage, the ballot access rules, the rules for postal voting, the opening hours of the polling places) would have to be harmonized all over the USA, leading to a very powerful central election authority.
>>
>> This property is desirable especially for the elections to the National Conventions for the nominations of the presidential candidates. Here, the election rules and the set of candidates differ significantly from state to state.

To combine the Schulze method with the Electoral College without losing any of its purposes, we recommend that, for each pair of candidates a and b separately, we should determine, how many electoral votes $N_{electors}[a,b]$ candidate a would get and how many electoral votes $N_{electors}[b,a]$ candidate b would get when only these two candidates were running. We then apply the Schulze method to the matrix $N_{electors}$.

So we recommend the following method:

Stage 1:

Suppose A is the set of candidates who are running in at least one state.

Suppose $A_X \subseteq A$ is the set of candidates who are running in state X.

For $a,b \in A_X$: $N_X[a,b] \in \mathbb{N}_0$ is the number of voters in state X who strictly prefer candidate a to candidate b.

Stage 2:

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “smaller_or_equal(y)” is the largest integer that is smaller than or equal to y. In other words: “smaller_or_equal(y)” is that integer $z \in \mathbb{N}_0$ with $z \leq y < (z + 1)$.

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “strictly_smaller(y)” is the largest integer that is strictly smaller than y. In other words: “strictly_smaller(y)” is that integer $z \in \mathbb{N}_0$ with $z < y \leq (z + 1)$.

Suppose $E_X \in \mathbb{N}$ is the number of electors of state X.

Suppose:

(a) $F_X[a,b] := E_X$,

if $\{ a \in A_X$ and $b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] > N_X[b,a] = 0 \}$.

(b) $F_X[a,b] := 0$,

if $\{ a \notin A_X$ and $b \in A_X \}$ or $\{ a,b \in A_X$ and $N_X[b,a] > N_X[a,b] = 0 \}$.

(c) $F_X[a,b] := E_X / 2$,

if $\{ a,b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] = N_X[b,a] \}$.

(d) $F_X[a,b] := 0.01 \cdot \text{smaller\_or\_equal}\left(\dfrac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,

if $a,b \in A_X$ and $N_X[a,b] > N_X[b,a] > 0$.

(e) $F_X[a,b] := 0.01 \cdot \text{strictly\_smaller}\left(\dfrac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,

if $a,b \in A_X$ and $N_X[b,a] > N_X[a,b] > 0$.

$N_{electors}[a,b] := \sum_X F_X[a,b]$.

Stage 3:

The Schulze method, as defined in section 2.2, is applied to $N_{electors}$.

Suppose the Schulze method is used for presidential primaries. Suppose some candidate g withdraws and doesn’t take part in the remaining primaries. Then candidate g is not removed from the pairwise matrix. Rather he is treated as described at stage 2 (a) – (c). This regulation is necessary because removing a loser can still change the winner.

9. Proportional Representation by the Single Transferable Vote

The term “Proportional Representation by the Single Transferable Vote” (STV) refers to preferential multi-winner election methods where the winning alternatives represent the electorate in a proportional manner. What exactly “in a proportional manner” means in this context is debatable and will be discussed in section 9.4.

A is a finite and non-empty set of alternatives. $M \in \mathbb{N}$ with $0 < M < \infty$ is the number of seats. $C \in \mathbb{N}$ with $M < C < \infty$ is the number of alternatives. $N \in \mathbb{N}$ with $0 < N < \infty$ is the number of voters.

A_M is the set of the $(C!)/((M!)\cdot((C–M)!))$ possible ways to choose M different alternatives from the set A. The elements of A_M are indicated with *wedding* letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ...

Input of an STV method is a profile, as defined in section 2.1. Output of an STV method is a subset $\emptyset \neq \mathcal{S}_M \subseteq A_M$ of potential winning sets.

9.1. Schulze STV

In Schulze STV, we only compare every set of M alternatives with every other set of M alternatives that differs in exactly one alternative.

There are $(C!)/((M!)\cdot(C–M)!)$ sets of exactly M alternatives.

There are $(C!)/(((M+1)!)\cdot((C–M–1)!))$ possible $(M+1)$-way contests. Each $(M+1)$-way contest leads to $M\cdot(M–1)$ links in that digraph where each node represents a set of M alternatives. See e.g. page 245.

So we have a digraph with $(C!)/((M!)\cdot(C–M)!)$ nodes and $M\cdot(M–1)\cdot(C!)/(((M+1)!)\cdot((C–M–1)!))$ links. This digraph is strongly connected. (A digraph is *strongly connected* : $\Leftrightarrow$ For every pair of two different nodes $\mathfrak{A}$ and $\mathfrak{B}$, there is a directed path from node $\mathfrak{A}$ to node $\mathfrak{B}$ and a directed path from node $\mathfrak{B}$ to node $\mathfrak{A}$.) We then apply the Schulze method, as defined in section 2.2, to this digraph. This works because, for the proof in section 4.1, it is sufficient that the digraph, that the Schulze method is applied to, is strongly connected. It is not necessary that this digraph is complete.

Schulze STV is motivated by the fact that we want a generalization of the Condorcet criterion from single-winner elections to multi-winner elections that is as strong as possible (section 9.3), so that the possibility, that an additional alternative changes the result of the election without being elected, is minimized. In section 10.3, we will see that the Condorcet criterion, that we get by this manner, is so strong that we almost always have M Condorcet winners or, at least, $(M–1)$ Condorcet winners.

9.1.1. Proportional Completion

Proportional completion means that non-linear individual orders are completed to linear orders in such a manner that, for each set of alternatives, the proportions of the individual orders, restricted to these alternatives, are not changed.

Example: Suppose a voter is indifferent between alternative a and alternative b. Suppose of the other voters $X_1 = 56$ strictly prefer alternative a to alternative b and $X_2 = 44$ strictly prefer alternative b to alternative a, then this voter is replaced by $X_1/(X_1+X_2) = 0.56$ voters who rank these alternatives $a \succ_v b$ and by $X_2/(X_1+X_2) = 0.44$ voters who rank these alternatives $b \succ_v a$ and who rank the other alternatives in the same manner as the original voter did.

Basic idea behind proportional completion is that, on the one side, adding a voter who is indifferent between all alternatives, that have chances to win, should not change the result of the election as this additional voter doesn’t add new information. On the other side, the definition for the strengths of the links between sets of alternatives (section 9.1.2) requires that each voter casts a linear order.

The following 3 stages give a precise definition for proportional completion.

Stage 1:

W shall be the proportional completion of V. $\rho(w) \in \mathbb{R}$ shall be the weight of voter $w \in W$. Then we start with

(9.1.1.1) $W := V$.

(9.1.1.2) $\forall\, w \in W$: $\rho(w) := 1$.

Stage 2:

Suppose there is a voter $w \in W$ and a set of alternatives $f_1,...,f_n \in A$ with

(9.1.1.3) $n > 1$.

(9.1.1.4) $\forall\, f_i,f_j \in \{f_1,...,f_n\}: f_i \approx_w f_j$.

(9.1.1.5) $\forall\, f_i \in \{f_1,...,f_n\}\ \forall\, e \in A \setminus \{f_1,...,f_n\}: f_i \not\approx_w e$.

Suppose $X \in \mathbb{N}_0$ is the number of voters $v \in V$ with

(9.1.1.6) $\exists\, f_i,f_j \in \{f_1,...,f_n\}: f_i \not\approx_v f_j$.

Case 1: $X > 0$.

For each voter $v \in V$ with (9.1.1.6), a voter u is added to W with

(9.1.1.7) $\forall\, g,h \in A \setminus \{f_1,...,f_n\}: g \succ_w h \Leftrightarrow g \succ_u h$.

(9.1.1.8) $\forall\, f_i \in \{f_1,...,f_n\}\ \forall\, g \in A \setminus \{f_1,...,f_n\}: g \succ_w f_i \Leftrightarrow g \succ_u f_i$.

(9.1.1.9) $\forall\, f_i \in \{f_1,...,f_n\}\ \forall\, h \in A \setminus \{f_1,...,f_n\}: f_i \succ_w h \Leftrightarrow f_i \succ_u h$.

(9.1.1.10) $\forall\, f_i,f_j \in \{f_1,...,f_n\}: f_i \succ_v f_j \Leftrightarrow f_i \succ_u f_j$.

(9.1.1.11) $\rho(u) := \rho(w) / X$.

Case 2: $X = 0$.

For each of the $n!$ possible permutations $\{\sigma(1),...,\sigma(n)\}$ of $\{1,...,n\}$, a voter u is added to W with (9.1.1.7) – (9.1.1.9) and

(9.1.1.12) $\forall\, f_i,f_j \in \{f_1,...,f_n\}: \sigma(i) > \sigma(j) \Leftrightarrow f_i \succ_u f_j$.

(9.1.1.13) $\rho(u) := \rho(w) / (n!)$.

After all these voters u have been added to W, the original voter w is removed from W.

Stage 3:

Stage 2 is repeated until $a \not\approx_w b\ \forall\, a \in A\ \forall\, b \in A \setminus \{a\}\ \forall\, w \in W$.

So in each iteration of proportional completion, we look whether there is still a voter who casts a non-linear order. When there is still such a voter, then we take a voter $w \in W$ and a set of alternatives $\emptyset \neq \{f_1,...,f_n\} \subseteq A$ (with $n > 1$) where voter w is indifferent between all the alternatives in $\{f_1,...,f_n\}$ [see (9.1.1.4)] and different between any alternative in $\{f_1,...,f_n\}$ and any alternative in $A \setminus \{f_1,...,f_n\}$ [see (9.1.1.5)]. We then look how those voters, who are not indifferent between all the alternatives in $\{f_1,...,f_n\}$ [see (9.1.1.6)], rank the alternatives in $\{f_1,...,f_n\}$. Voter w is then replaced, in a proportional manner [see (9.1.1.11)], by voters who rank the alternatives in $A \setminus \{f_1,...,f_n\}$ in the same order as voter w did [see (9.1.1.7) – (9.1.1.9)] and who rank the alternatives in $\{f_1,...,f_n\}$ in the same order as the other voters do [see (9.1.1.10)].

9.1.2. Links between Sets of Winners

Basic idea for the definition of the strength of some set of M alternatives $\{a_1,...,a_M\} \subset A$ against some alternative $b \in A \setminus \{a_1,...,a_M\}$ is that a defeat of alternative a against alternative b of strength $N[a,b]$ is single-winner elections corresponds to a situation in M-seat elections where each of the alternatives $\{a_1,...,a_M\}$ has a “separate quota” against alternative b of strength $N[\{a_1,...,a_M\};b]$. See (9.1.2.5) – (9.1.2.6).

W is the proportional completion of V. $\rho(w) \in \mathbb{R}$ is the weight of voter $w \in W$. N_W is the number of voters in W.

Suppose $\{a_1,...,a_M\} \subset A$ and $\{a_1,...,a_{(M-1)},b\} \subset A$ are two sets of alternatives that differ in exactly one alternative. Then the strength $N[\{a_1,...,a_M\};b] \in \mathbb{R}$ of the link from $\{a_1,...,a_M\}$ to $\{a_1,...,a_{(M-1)},b\}$ is defined as follows:

> $N[\{a_1,...,a_M\};b] \in \mathbb{R}$ is the largest value such that there is a $t \in \mathbb{R}^{(N_W \times M)}$ such that

(9.1.2.1) $\quad \forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $t_{ij} \geq 0$.

(9.1.2.2) $\quad \forall\, i \in \{1,...,N_W\}$: $\displaystyle\sum_{j=1}^{M} t_{ij} \leq \rho(i)$.

(9.1.2.3) $\quad \forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $b \succ_i a_j \Rightarrow t_{ij} = 0$.

(9.1.2.4) $\quad \forall\, j \in \{1,...,M\}$: $\displaystyle\sum_{i=1}^{N_W} t_{ij} \geq N[\{a_1,...,a_M\};b]$.

So the strength $N[\{a_1,...,a_M\};b]$ of the link from set $\{a_1,...,a_M\}$ to set $\{a_1,...,a_{(M-1)},b\}$ is the largest number such that the electorate can be divided into M+1 disjoint sets $T_1,...,T_{(M+1)}$ such that:

(9.1.2.5) $\quad \forall\, j \in \{1,...,M\}$: Every voter in T_j prefers alternative a_j to alternative b.

(9.1.2.6) $\quad \forall\, j \in \{1,...,M\}$: The total weight of the voters in T_j is at least $N[\{a_1,...,a_M\};b]$.

Basic idea behind this definition for the strength of a link is the following:

In multi-winner elections, it is a useful strategy for a voter not to give a needlessly good preference to an alternative that wins with certainty even without this voter’s vote. By using this strategy, this voter doesn’t waste his vote (to alternatives that win with certainty even without this voter’s vote) so that his vote has more impact on which of those alternatives, that are less certain of getting elected, actually get elected. This strategy is called “free riding” (Schulze, 2004).

When the voters have understood this strategic loophole well, then the order, in which the individual voter ranks the strong alternatives relatively to each other, doesn’t say anything anymore about the sincere opinion of this voter about these alternatives, it only says something about how strong this voter believes these alternatives are relatively to each other. So the order, in which the individual voter ranks the strong alternatives relatively to each other, doesn’t contain any information and should, therefore, have no impact on the result of the election.

In the above definition for the strength $N[\{a_1,...,a_M\};b]$ of the link from set $\{a_1,...,a_M\}$ to set $\{a_1,...,a_{(M-1)},b\}$, this strength does not depend on the order in which the individual voter ranks the alternatives $\{a_1,...,a_M\}$ relatively to each other, it only depends on which alternatives of $\{a_1,...,a_M\}$ are preferred to alternative b [see (9.1.2.5) and (9.1.2.6)]. As, in Schulze STV, the strengths of links against strong alternatives have no impact on the result of the election [see section 9.3], the above definition for the strengths of links guarantees that Schulze STV is invulnerable to free riding.

9.1.3. Definition of Schulze STV

Suppose $\succ_{D1}$ and $\succ_{D2}$ are two binary relations that each satisfy (2.1.1) – (2.1.3).

Stage 1:

We calculate the Schulze single-winner ranking O_1 on A, as defined in section 5, with $\succ_{D1}$.

Stage 2:

Proportional completion is used to complete V to W.

Stage 3:

A *path* from set $\mathfrak{X} \in A_M$ to set $\mathfrak{Y} \in A_M \setminus \{\mathfrak{X}\}$ is a sequence of sets $\mathfrak{C}(1),...,\mathfrak{C}(n) \in A_M$ with the following properties:

1. $\mathfrak{X} \equiv \mathfrak{C}(1)$.
2. $\mathfrak{Y} \equiv \mathfrak{C}(n)$.
3. $2 \leq n < \infty$.
4. For all $i = 1,...,(n–1)$: $\mathfrak{C}(i)$ and $\mathfrak{C}(i+1)$ differ in exactly one alternative. That means: $|\ \mathfrak{C}(i) \cap \mathfrak{C}(i+1)\ | = M - 1$ and $|\ \mathfrak{C}(i) \cup \mathfrak{C}(i+1)\ | = M + 1$.

The *strength* of the path $\mathfrak{C}(1),...,\mathfrak{C}(n)$ is

$$\begin{aligned}\min_{D2}\ \{\ &(N[\{a_1,...,a_{(M-1)},b\};c],\ N[\{a_1,...,a_{(M-1)},c\};b]) \\ &\text{with } \{a_1,...,a_{(M-1)}\} := \mathfrak{C}(i) \cap \mathfrak{C}(i+1), \\ &b := \mathfrak{C}(i) \setminus \mathfrak{C}(i+1), \text{ and } c := \mathfrak{C}(i+1) \setminus \mathfrak{C}(i) \\ &\quad |\ i = 1,...,(n-1)\ \}.\end{aligned}$$

In other words: The strength of a path is the strength of its weakest link.

$$\begin{aligned}P_{D2}[\mathfrak{A},\mathfrak{B}] := \max_{D2}\ \{\ &\\ \min_{D2}\ \{\ &(N[\{a_1,...,a_{(M-1)},b\};c],\ N[\{a_1,...,a_{(M-1)},c\};b]) \\ &\text{with } \{a_1,...,a_{(M-1)}\} := \mathfrak{C}(i) \cap \mathfrak{C}(i+1), \\ &b := \mathfrak{C}(i) \setminus \mathfrak{C}(i+1), \text{ and } c := \mathfrak{C}(i+1) \setminus \mathfrak{C}(i) \\ &\quad |\ i = 1,...,(n-1)\ \} \\ &\quad |\ \mathfrak{C}(1),...,\mathfrak{C}(n) \text{ is a path from set } \mathfrak{A} \text{ to set } \mathfrak{B}\ \}.\end{aligned}$$

In other words: $P_{D2}[\mathfrak{A},\mathfrak{B}] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from set $\mathfrak{A} \in A_M$ to set $\mathfrak{B} \in A_M \setminus \{\mathfrak{A}\}$.

(9.1.3.1) The binary relation O_M on A_M is defined as follows:

$$\mathfrak{A}\mathfrak{B} \in O_M :\Leftrightarrow P_{D2}[\mathfrak{A},\mathfrak{B}] \succ_{D2} P_{D2}[\mathfrak{B},\mathfrak{A}].$$

(9.1.3.2) $\mathcal{S}_M := \{\ \mathfrak{A} \in A_M\ |\ \forall\ \mathfrak{B} \in A_M \setminus \{\mathfrak{A}\}: \mathfrak{B}\mathfrak{A} \notin O_M\ \}$ is the *set of potential winning sets*.

Stage 4:

For all $\mathfrak{A},\mathfrak{B} \in \mathcal{S}_M$: Suppose there is an alternative $a \in \mathfrak{A} \setminus \mathfrak{B}$ with $ab \in \mathcal{O}_1$ for every alternative $b \in \mathfrak{B} \setminus \mathfrak{A}$, then the set $\mathfrak{A}$ *disqualifies* the set $\mathfrak{B}$.

The winning set of Schulze STV is that set $\mathfrak{A} \in \mathcal{S}_M$ that is not disqualified by some other set $\mathfrak{B} \in \mathcal{S}_M$.

9.2. Example A53

To illustrate Schulze STV, we will use a rather large example because smaller examples usually don't address all aspects of an STV election. We will use example A53 from Tideman's database. This example is analysed in great detail by Tideman (2000). Example A53 consists of V = 460 voters and C = 10 alternatives running for M = 4 seats.

Example A53 is interesting because the Newland-Britton (1997) method, the Meek (1969, 1970; Hill, 1987) method, and the Warren (1994) method each find a different set of winners. The Newland-Britton method chooses *a*, *b*, *g*, and *j*. The Meek method chooses *a*, *d*, *g*, and *j*. The Warren method chooses *a*, *f*, *g*, and *j*.

Example A53 is decribed in the following table 9.2.1. For example, row 233 says that voter 233 gives a "1" to alternative *b*, a "2" to alternative *c*, a "3" to alternative *d*, a "4" to alternative *a*, and a "5" to alternative *j*. Voter 233 doesn't rank any of the other alternatives.

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
1	1	-	-	-	-	4	3	2	-	-
2	1	2	4	5	3	9	6	10	8	7
3	2	6	10	7	3	8	5	9	1	4
4	-	-	-	-	-	-	-	-	-	1
5	-	-	-	-	-	-	-	-	-	1
6	3	-	-	-	5	4	6	7	2	1
7	4	-	3	-	-	-	-	-	2	1
8	3	-	1	-	-	-	-	-	4	2
9	2	-	1	-	-	-	-	-	-	-
10	3	-	-	-	-	-	-	2	-	1
11	-	5	-	-	1	4	2	-	3	6
12	-	4	5	-	1	-	2	3	-	-
13	7	9	6	10	1	5	3	4	8	2
14	4	5	3	9	1	10	2	6	8	7
15	-	-	-	-	2	-	3	-	1	4
16	-	-	4	-	2	-	3	-	1	-
17	2	-	5	-	1	-	4	-	6	3
18	-	-	-	-	1	4	2	-	-	3
19	3	-	-	6	-	4	5	1	-	2
20	4	-	-	5	-	2	3	-	-	1
21	4	9	7	5	8	2	10	6	3	1
22	4	7	3	6	8	2	10	5	9	1
23	4	-	-	6	3	2	-	5	-	1
24	-	5	-	4	-	3	-	-	2	1
25	-	-	-	4	-	3	-	-	2	1
26	4	10	9	8	7	3	5	6	2	1
27	4	-	-	6	-	3	-	5	2	1
28	3	4	-	2	-	-	-	5	-	1
29	3	-	-	2	-	-	-	-	-	1
30	8	9	7	2	3	4	5	6	10	1
31	5	7	6	2	3	4	10	9	8	1
32	10	8	4	2	3	9	7	5	6	1
33	4	6	-	2	5	3	-	-	-	1
34	-	-	-	2	-	-	3	-	-	1
35	3	9	7	2	4	8	6	10	5	1
36	1	5	2	-	-	-	3	-	-	4
37	1	7	5	3	9	6	8	4	10	2
38	1	7	5	6	3	2	8	9	10	4
39	1	2	4	3	10	9	5	8	7	6
40	1	-	-	3	6	2	-	4	-	5
41	1	-	-	-	-	-	2	-	-	-
42	1	8	5	2	4	3	10	9	7	6
43	1	-	-	2	3	5	-	-	4	-
44	1	2	3	5	7	4	8	9	10	6
45	1	2	7	4	6	5	9	10	8	3
46	1	2	3	5	4	6	10	11	12	7
47	1	3	8	7	6	5	4	2	9	10
48	1	7	6	4	5	2	8	10	9	3
49	1	3	7	6	10	4	9	5	8	2
50	1	8	10	4	7	2	3	9	6	5
51	1	-	6	-	2	3	-	7	5	-
52	1	3	6	10	7	9	5	2	8	4
53	1	10	4	9	7	2	5	8	3	6
54	1	-	-	-	-	2	-	3	-	4
55	1	2	-	-	-	-	3	-	-	4
56	4	5	-	-	6	-	-	3	2	1
57	4	5	6	10	8	9	7	3	2	1
58	4	5	9	6	7	8	10	2	3	1
59	6	3	5	7	10	4	8	2	9	1
60	10	3	4	9	5	6	8	2	7	1

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
61	3	4	2	7	8	9	10	5	6	1
62	7	3	6	9	2	8	5	10	4	1
63	4	3	7	9	2	6	10	5	8	1
64	5	3	10	7	2	9	8	6	4	1
65	-	1	2	-	-	-	-	-	-	3
66	-	-	1	6	2	-	3	-	4	5
67	-	-	4	3	1	-	2	-	-	-
68	2	9	8	5	1	6	3	7	10	4
69	6	3	7	9	2	8	1	10	4	5
70	4	9	6	10	3	7	1	8	5	2
71	3	9	5	6	4	8	1	7	10	2
72	2	-	-	-	-	-	1	4	-	3
73	7	4	8	5	9	6	1	2	10	3
74	9	10	8	5	7	6	1	2	3	4
75	-	-	-	2	3	-	1	-	-	-
76	7	8	10	3	2	6	1	9	4	5
77	4	3	2	-	-	-	1	-	-	-
78	3	7	4	5	6	8	1	9	10	2
79	5	10	6	7	2	8	1	9	3	4
80	2	5	4	6	7	10	1	8	9	3
81	-	-	-	-	-	-	1	-	-	2
82	-	-	-	3	-	-	1	2	-	-
83	2	4	3	9	10	8	1	5	7	6
84	4	7	2	6	5	8	1	9	10	3
85	-	2	-	-	3	-	1	-	-	-
86	2	5	-	-	4	-	1	-	-	3
87	5	-	-	2	4	-	1	3	-	6
88	-	-	-	-	-	-	1	-	2	-
89	2	-	4	-	-	-	1	-	-	3
90	7	3	4	2	9	6	1	8	10	5
91	2	8	5	7	6	3	1	10	9	4
92	7	10	6	9	5	4	1	8	3	2
93	-	2	-	-	-	-	1	3	-	-
94	3	-	-	2	-	-	1	-	-	-
95	-	-	-	-	-	3	1	-	2	4
96	-	-	-	2	-	3	1	4	-	-
97	-	-	-	-	-	2	1	-	3	4
98	6	10	8	9	7	5	1	4	2	3
99	3	-	-	2	-	-	1	-	-	-
100	5	-	-	4	-	-	1	3	-	2
101	4	-	-	3	5	-	1	-	-	2
102	3	-	-	-	-	2	1	-	5	4
103	4	3	-	2	-	-	1	-	5	6
104	3	-	-	4	-	2	1	-	6	5
105	-	4	-	2	-	3	1	-	-	-
106	3	-	4	-	5	-	1	6	-	2
107	10	5	3	2	4	9	1	7	8	6
108	3	-	-	1	-	-	-	-	-	2
109	-	3	2	1	-	4	-	-	-	-
110	2	-	-	1	-	-	-	-	4	3
111	-	-	-	1	-	-	-	-	-	2
112	5	4	-	1	-	-	-	3	-	2
113	-	-	-	1	-	-	2	-	-	-
114	6	5	10	1	3	7	2	8	9	4
115	-	-	-	1	4	-	-	-	3	2
116	2	-	-	1	-	-	-	-	-	3
117	-	4	-	1	-	3	-	-	-	2
118	4	3	-	1	-	-	2	-	-	-
119	2	5	6	1	3	4	7	10	9	8
120	-	-	-	1	-	-	2	-	-	-

Table 9.2.1 (part 1 of 4): Example A53

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
121	2	-	-	1	-	3	-	-	-	4
122	-	-	-	1	-	-	2	-	-	3
123	3	4	-	1	-	-	2	-	-	-
124	-	-	-	1	-	-	-	-	-	-
125	-	-	-	1	-	2	4	-	3	-
126	-	-	-	1	-	-	-	-	-	-
127	4	5	7	1	2	9	6	8	3	10
128	3	-	-	1	-	-	-	2	-	4
129	2	3	-	1	-	-	5	-	-	4
130	3	10	6	1	4	8	7	9	5	2
131	2	-	4	1	-	5	-	-	-	3
132	-	-	-	1	-	-	-	3	-	2
133	-	-	-	1	2	3	-	-	-	4
134	3	2	7	1	6	9	10	5	8	4
135	2	5	6	1	-	3	4	-	-	-
136	5	4	8	1	6	9	7	3	2	10
137	2	9	5	1	3	10	8	6	4	7
138	-	-	-	1	-	2	-	3	-	4
139	-	-	-	1	2	3	-	-	-	-
140	9	7	8	1	2	6	3	10	5	4
141	3	4	6	1	7	9	2	10	8	5
142	5	6	7	1	2	9	3	10	4	8
143	3	9	6	1	10	4	2	7	8	5
144	-	4	-	1	5	3	-	-	2	-
145	-	-	-	1	-	-	-	-	-	-
146	2	6	9	1	8	5	10	3	7	4
147	2	-	3	1	-	-	-	-	4	5
148	-	-	-	1	-	2	-	-	-	3
149	2	-	-	1	4	-	-	-	-	3
150	3	6	5	2	7	10	8	9	1	4
151	8	6	7	4	5	10	9	3	1	2
152	-	-	-	3	-	-	4	1	-	2
153	7	-	1	3	6	5	4	-	-	2
154	-	-	1	2	-	-	-	-	-	-
155	-	5	2	3	-	4	-	-	1	6
156	-	-	2	3	5	-	4	-	1	6
157	5	4	9	2	1	6	7	8	10	3
158	4	10	5	2	1	6	3	9	8	7
159	-	-	-	2	1	-	-	-	-	-
160	2	-	-	3	1	-	5	-	-	4
161	2	9	7	5	1	8	6	4	10	3
162	-	-	1	3	2	4	-	-	-	-
163	3	8	9	4	10	1	5	2	6	7
164	-	-	6	4	3	1	-	5	-	2
165	2	8	7	3	4	1	5	9	10	6
166	8	6	5	3	10	1	4	7	9	2
167	5	-	6	3	-	1	-	4	-	2
168	6	8	5	7	4	1	9	3	10	2
169	4	-	-	3	-	1	-	-	5	2
170	6	8	7	2	9	1	10	5	4	3
171	2	-	-	3	-	1	-	-	4	-
172	8	9	6	2	4	1	3	10	5	7
173	2	-	-	5	3	1	4	-	-	-
174	-	5	-	3	4	1	-	-	-	2
175	2	6	7	3	5	1	8	10	9	4
176	9	10	3	8	2	1	4	7	5	6
177	4	-	-	2	-	1	3	-	-	-
178	9	6	4	5	2	3	10	8	1	7
179	5	-	4	-	-	3	-	2	-	1
180	-	-	4	-	6	3	5	2	-	1

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
181	3	-	-	-	-	5	6	2	4	1
182	3	-	-	-	-	4	-	2	-	1
183	4	-	-	-	-	2	-	3	-	1
184	-	-	-	-	-	2	3	-	-	1
185	4	-	-	-	-	2	3	-	-	1
186	3	-	-	-	-	2	4	-	-	1
187	3	4	5	8	7	2	6	9	10	1
188	7	8	9	10	3	6	4	2	5	1
189	-	4	-	-	2	3	-	-	-	1
190	4	9	7	10	2	5	8	3	6	1
191	3	5	8	6	10	2	7	9	4	1
192	10	5	6	9	4	3	7	8	2	1
193	6	4	-	5	-	3	-	7	2	1
194	5	8	6	10	4	3	9	7	2	1
195	-	7	-	-	5	3	4	6	2	1
196	5	10	9	4	6	7	3	8	1	2
197	-	-	-	-	3	-	2	-	1	-
198	-	-	-	-	-	3	2	-	1	4
199	-	-	-	-	-	-	2	1	-	-
200	3	9	1	5	10	6	2	4	8	7
201	2	7	1	5	6	4	3	8	9	10
202	2	6	1	7	9	10	3	8	5	4
203	-	5	1	-	-	-	2	3	4	-
204	-	-	1	-	4	-	3	-	5	2
205	9	10	1	8	7	3	2	5	4	6
206	2	-	3	5	-	-	4	1	-	-
207	4	-	-	-	3	-	-	2	-	1
208	4	-	-	-	3	-	-	2	-	1
209	-	-	-	-	2	-	-	-	-	1
210	-	-	-	-	2	-	-	3	4	1
211	2	6	5	8	3	9	7	1	10	4
212	3	-	4	-	-	2	1	-	-	-
213	-	-	-	-	-	-	1	2	3	-
214	-	4	-	-	-	2	1	3	-	-
215	-	-	5	-	6	4	1	-	3	2
216	5	4	6	-	8	2	1	7	-	3
217	8	10	3	6	7	2	1	4	9	5
218	-	-	-	3	-	2	1	-	4	5
219	3	4	5	10	6	9	1	8	7	2
220	2	6	10	8	7	4	1	5	9	3
221	-	3	-	2	-	4	1	-	-	-
222	-	-	-	3	4	-	-	-	2	1
223	3	-	5	4	-	-	-	2	-	1
224	4	-	-	3	-	-	-	2	-	1
225	3	-	2	4	-	-	-	-	-	1
226	4	9	2	6	5	7	8	3	10	1
227	3	7	4	6	5	9	8	2	10	1
228	5	4	-	3	2	-	-	-	-	1
229	3	1	-	-	-	5	4	-	6	2
230	2	1	-	-	-	-	4	-	3	-
231	2	1	-	-	-	3	-	-	4	-
232	-	1	-	-	-	-	3	-	-	2
233	4	1	2	3	-	-	-	-	-	5
234	6	1	5	7	4	8	2	9	10	3
235	9	1	5	10	4	2	6	8	7	3
236	4	1	5	9	3	6	8	7	10	2
237	-	1	-	-	3	-	2	-	-	-
238	2	1	3	-	-	4	5	-	-	-
239	-	1	-	-	2	-	-	3	-	4
240	-	1	-	-	-	-	-	-	-	-

Table 9.2.1 (part 2 of 4): Example A53

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
241	3	1	7	2	8	9	4	10	6	5
242	-	1	-	-	2	-	3	4	5	6
243	2	1	9	6	5	7	10	3	8	4
244	7	1	8	3	6	4	5	9	10	2
245	5	1	6	7	2	10	3	9	8	4
246	2	1	8	5	9	6	10	7	3	4
247	2	1	3	10	4	5	7	6	8	9
248	-	1	-	-	2	4	-	3	-	-
249	6	1	4	8	7	2	5	10	9	3
250	2	1	9	10	7	6	4	5	8	3
251	4	1	5	6	-	-	2	7	-	3
252	-	1	-	-	-	2	-	-	3	4
253	-	1	5	-	2	-	3	-	-	4
254	3	1	5	2	8	4	10	7	9	6
255	6	1	5	7	4	8	3	9	2	10
256	5	1	6	3	2	9	4	7	8	10
257	9	1	6	5	3	10	2	4	7	8
258	6	1	3	5	4	10	2	8	9	7
259	-	1	-	-	-	2	-	-	3	4
260	-	1	-	-	4	3	5	-	-	2
261	-	1	4	-	-	2	-	-	-	3
262	-	1	-	-	2	-	-	3	4	-
263	-	1	-	-	-	4	2	-	-	3
264	5	1	9	4	2	6	10	7	8	3
265	2	1	8	7	6	5	9	10	3	4
266	3	1	7	6	9	5	4	8	10	2
267	9	2	8	3	7	10	4	6	1	5
268	-	3	-	4	-	5	-	1	-	2
269	-	3	2	-	6	-	5	1	-	4
270	7	4	2	10	5	9	6	1	3	8
271	6	4	3	10	5	7	9	1	8	2
272	3	2	1	-	-	-	-	-	4	-
273	-	2	1	-	-	3	-	-	-	4
274	9	5	8	7	2	6	10	1	3	4
275	4	2	8	7	1	10	9	5	3	6
276	-	2	-	-	1	-	-	-	-	-
277	4	5	6	8	3	9	10	1	7	2
278	6	3	5	10	1	4	9	7	8	2
279	2	6	9	10	1	7	8	3	4	5
280	6	4	-	-	3	5	-	1	-	2
281	7	2	8	6	4	9	3	10	5	1
282	3	2	-	-	-	-	4	-	-	1
283	5	2	8	4	3	6	10	7	9	1
284	7	2	9	8	3	10	4	5	6	1
285	-	2	-	3	-	-	-	4	-	1
286	5	2	10	7	4	3	8	6	9	1
287	4	2	6	5	3	8	7	10	9	1
288	2	3	9	5	10	6	7	4	8	1
289	2	4	-	6	3	-	-	5	-	1
290	2	4	-	-	5	6	3	-	-	1
291	2	3	-	-	-	-	5	4	-	1
292	2	3	10	4	9	5	6	7	8	1
293	2	-	-	-	3	-	4	5	-	1
294	2	-	-	-	-	4	-	3	-	1
295	2	3	4	8	7	10	5	6	9	1
296	2	-	-	-	-	3	-	4	-	1
297	2	-	-	-	-	3	4	-	-	1
298	2	-	3	4	-	5	-	-	-	1
299	2	-	-	-	3	-	-	-	4	1
300	2	5	4	3	6	-	-	-	-	1

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
301	2	-	-	-	-	-	4	-	3	1
302	2	10	3	4	6	9	5	7	8	1
303	2	-	5	-	-	3	-	-	4	1
304	2	3	6	7	8	4	9	5	10	1
305	2	5	-	-	-	-	-	4	3	1
306	2	3	-	-	-	-	-	4	-	1
307	2	3	10	8	7	4	9	5	6	1
308	2	-	-	-	-	3	-	2	3	1
309	2	3	-	-	-	-	4	-	-	1
310	2	9	4	5	6	7	3	10	8	1
311	2	3	6	4	-	-	-	-	5	1
312	2	-	-	4	-	-	-	-	3	1
313	2	10	3	9	6	5	8	7	4	1
314	2	-	-	-	-	3	-	4	-	1
315	2	3	5	8	7	6	9	4	10	1
316	2	-	-	3	-	-	4	-	-	1
317	2	3	7	6	8	4	9	5	10	1
318	2	4	-	3	-	-	5	-	-	1
319	2	-	5	3	4	-	6	-	-	1
320	2	3	6	10	5	4	7	8	9	1
321	2	4	7	8	5	9	10	3	6	1
322	2	-	-	5	3	-	4	-	-	1
323	-	-	4	-	1	-	-	-	3	2
324	-	-	-	-	1	-	-	3	4	2
325	-	-	-	-	1	-	-	-	-	-
326	-	-	-	-	1	-	-	-	-	2
327	2	-	3	-	1	-	-	-	-	-
328	4	-	5	3	-	-	1	-	-	2
329	5	-	2	4	-	3	1	-	-	-
330	-	-	-	2	-	-	1	-	-	3
331	-	-	-	2	-	-	1	4	-	3
332	-	-	3	2	4	5	1	-	-	-
333	3	5	-	-	-	4	1	6	-	2
334	5	-	6	4	2	-	1	-	3	-
335	2	3	-	-	-	-	1	-	-	-
336	-	3	2	-	-	-	1	-	-	-
337	6	3	7	2	4	10	1	8	9	5
338	-	4	5	-	-	2	1	-	3	-
339	-	3	-	4	-	-	1	2	-	-
340	-	-	-	-	-	-	1	-	-	-
341	8	9	7	2	3	4	1	10	5	6
342	2	3	5	6	9	10	1	4	8	7
343	2	7	3	4	9	8	1	10	6	5
344	5	4	9	3	8	10	1	7	2	6
345	-	-	-	-	-	-	1	-	-	-
346	2	-	-	4	3	-	1	6	7	5
347	4	9	7	10	3	2	1	8	6	5
348	9	8	7	2	10	6	1	5	4	3
349	3	-	-	-	-	-	2	-	-	1
350	-	4	-	-	-	-	2	-	3	1
351	-	-	-	-	-	-	2	3	4	1
352	-	-	-	-	-	-	2	-	-	1
353	3	-	-	-	4	-	2	-	-	1
354	3	4	8	7	10	9	2	6	5	1
355	3	-	-	-	-	-	2	-	4	1
356	-	4	-	3	-	-	2	-	-	1
357	5	8	7	3	10	4	2	9	6	1
358	-	-	-	3	4	-	2	-	-	1
359	-	4	-	3	-	-	2	-	-	1
360	-	4	-	-	-	3	2	-	-	1

Table 9.2.1 (part 3 of 4): Example A53

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
361	4	3	-	-	-	-	2	-	-	1
362	-	3	-	4	-	-	2	-	-	1
363	-	-	-	-	-	-	3	-	2	1
364	-	-	-	-	4	-	2	3	1	-
365	3	7	8	10	9	6	4	2	5	1
366	5	9	10	6	8	4	3	2	7	1
367	7	4	2	6	8	5	3	10	9	1
368	-	4	-	-	3	1	2	-	-	-
369	3	7	6	10	9	1	2	4	8	5
370	3	4	-	-	-	1	2	-	-	5
371	4	2	-	-	-	1	-	3	-	-
372	5	6	7	10	8	1	2	3	9	4
373	4	3	5	9	8	1	6	10	7	2
374	4	8	5	9	7	1	3	-	2	6
375	2	4	3	5	6	1	10	8	9	7
376	5	9	2	10	3	1	6	7	4	8
377	-	2	-	-	-	1	3	-	4	5
378	6	7	10	9	5	1	2	9	4	3
379	3	2	-	4	-	1	-	-	-	-
380	5	6	4	10	3	1	2	7	9	8
381	5	4	7	8	1	3	6	9	10	2
382	2	5	-	-	1	4	-	-	3	-
383	3	6	7	9	1	2	8	5	4	10
384	-	2	5	-	3	1	4	-	-	-
385	4	-	3	-	-	1	-	-	-	2
386	4	-	-	-	2	1	-	-	-	3
387	-	-	3	-	-	1	2	-	-	-
388	-	-	3	-	-	1	4	-	-	2
389	-	-	-	-	-	1	3	4	-	2
390	-	-	5	-	-	1	-	3	4	2
391	5	-	3	-	-	1	4	-	6	2
392	-	-	-	-	2	1	5	-	3	4
393	4	-	5	-	-	2	-	-	1	3
394	-	-	-	-	1	2	3	-	4	-
395	-	-	1	-	2	3	-	4	-	-
396	-	-	2	-	1	3	-	4	-	5
397	-	-	4	-	3	1	2	-	-	-
398	3	-	-	-	-	1	-	-	-	2
399	-	-	-	-	-	1	-	-	-	2
400	-	-	2	-	-	1	-	-	-	-
401	1	7	5	2	10	8	6	9	3	4
402	1	-	3	-	-	-	2	4	-	-
403	1	6	8	2	9	5	7	4	10	3
404	1	-	-	2	-	-	-	-	-	3
405	1	-	-	-	-	-	2	-	-	3
406	1	4	-	-	-	-	-	2	-	3
407	1	-	-	5	4	-	-	-	3	2
408	1	-	4	5	-	6	3	7	8	2
409	1	-	5	3	4	-	2	-	-	-
410	1	10	9	8	6	4	7	2	5	3

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
411	1	-	3	2	-	-	4	-	-	-
412	1	3	-	-	2	4	-	-	-	-
413	1	10	6	4	9	8	3	7	2	5
414	1	-	-	-	-	2	-	-	-	-
415	1	-	2	-	-	-	-	3	-	4
416	1	4	-	-	-	3	2	-	-	5
417	1	3	-	-	-	2	-	-	4	-
418	1	-	-	4	-	5	3	-	-	2
419	1	7	4	7	8	9	3	5	6	2
420	1	-	-	-	-	2	3	-	-	4
421	1	6	3	7	2	9	5	4	10	8
422	1	-	2	7	8	9	3	4	6	5
423	1	-	-	2	-	-	3	-	-	-
424	1	10	9	2	5	3	7	6	8	4
425	1	3	-	-	-	-	-	4	-	2
426	1	-	-	-	-	2	-	-	-	3
427	1	6	-	-	3	-	-	4	5	2
428	1	8	9	4	7	2	3	10	6	5
429	1	6	10	3	9	7	8	2	4	5
430	1	-	-	-	-	4	3	-	-	2
431	1	9	2	8	3	10	4	6	7	5
432	1	4	-	-	-	-	-	3	-	2
433	1	-	3	-	-	-	4	-	-	2
434	1	-	-	-	-	-	-	-	-	2
435	1	-	3	-	-	2	-	-	5	4
436	1	-	2	3	5	-	4	-	-	6
437	1	-	-	-	-	2	3	-	-	4
438	1	4	10	6	5	8	2	9	3	7
439	1	-	-	3	-	-	4	-	-	2
440	1	-	-	-	-	-	2	4	3	-
441	1	6	2	5	3	9	10	7	4	8
442	1	8	9	2	4	7	10	5	6	3
443	1	-	-	-	-	-	2	-	-	-
444	1	2	-	-	-	-	4	-	5	3
445	1	7	8	9	6	4	10	3	5	2
446	1	3	-	-	-	5	-	4	-	2
447	1	-	2	4	3	-	-	-	-	-
448	-	1	-	-	-	-	-	-	-	-
449	1	4	-	5	-	3	-	-	-	2
450	1	2	6	9	5	7	8	3	10	4
451	1	2	3	-	-	-	4	-	-	-
452	1	3	-	4	-	-	-	-	-	2
453	1	-	-	2	3	-	-	-	-	4
454	1	3	2	8	7	10	9	6	4	5
455	1	-	-	3	-	4	-	-	-	2
456	1	-	-	-	-	2	-	4	-	3
457	1	6	10	2	5	8	3	9	4	7
458	1	4	10	5	9	8	6	2	7	3
459	1	-	3	2	-	-	-	-	-	-
460	1	3	7	2	10	8	6	9	4	5

Table 9.2.1 (part 4 of 4): Example A53

9.2.1. Proportional Completion

We apply proportional completion separately for the calculation of each link. The strength of the links {*b*,*c*,*e*,*j*} → {*a*,*c*,*e*,*j*}, {*b*,*c*,*e*,*j*} → {*a*,*b*,*e*,*j*}, {*b*,*c*,*e*,*j*} → {*a*,*b*,*c*,*j*}, and {*b*,*c*,*e*,*j*} → {*a*,*b*,*c*,*e*} depends only on whether the individual voter strictly prefers the different candidates of the set {*b*,*c*,*e*,*j*} to candidate *a* or strictly prefers candidate *a* to the different candidates of the set {*b*,*c*,*e*,*j*} or is indifferent between the different candidates of the set {*b*,*c*,*e*,*j*} and candidate *a*. Therefore, the fact, that we apply proportional completion for every link separately, means that only 3^*C* = 81 possible voting patterns need to be considered. Table 9.2.1.1 lists these 81 possible voting patterns, where “1” means that a voter with this voting pattern strictly prefers this candidate to candidate *a*, a “2” means that this voter is indifferent between this candidate and candidate *a*, and a “3” means that this voter strictly prefers candidate *a* to this candidate.

Throughout section 9.2.1, w_j^i is the number of voters at stage *j* who are using voting pattern *i*.

voting pattern	*b*	*c*	*e*	*j*
#1	1	1	1	1
#2	1	1	1	2
#3	1	1	1	3
#4	1	1	2	1
#5	1	1	2	2
#6	1	1	2	3
#7	1	1	3	1
#8	1	1	3	2
#9	1	1	3	3
#10	1	2	1	1
#11	1	2	1	2
#12	1	2	1	3
#13	1	2	2	1
#14	1	2	2	2
#15	1	2	2	3
#16	1	2	3	1
#17	1	2	3	2
#18	1	2	3	3
#19	1	3	1	1
#20	1	3	1	2
#21	1	3	1	3
#22	1	3	2	1
#23	1	3	2	2
#24	1	3	2	3
#25	1	3	3	1
#26	1	3	3	2
#27	1	3	3	3
#28	2	1	1	1
#29	2	1	1	2
#30	2	1	1	3
#31	2	1	2	1
#32	2	1	2	2
#33	2	1	2	3
#34	2	1	3	1
#35	2	1	3	2
#36	2	1	3	3
#37	2	2	1	1
#38	2	2	1	2
#39	2	2	1	3
#40	2	2	2	1

voting pattern	*b*	*c*	*e*	*j*
#41	2	2	2	2
#42	2	2	2	3
#43	2	2	3	1
#44	2	2	3	2
#45	2	2	3	3
#46	2	3	1	1
#47	2	3	1	2
#48	2	3	1	3
#49	2	3	2	1
#50	2	3	2	2
#51	2	3	2	3
#52	2	3	3	1
#53	2	3	3	2
#54	2	3	3	3
#55	3	1	1	1
#56	3	1	1	2
#57	3	1	1	3
#58	3	1	2	1
#59	3	1	2	2
#60	3	1	2	3
#61	3	1	3	1
#62	3	1	3	2
#63	3	1	3	3
#64	3	2	1	1
#65	3	2	1	2
#66	3	2	1	3
#67	3	2	2	1
#68	3	2	2	2
#69	3	2	2	3
#70	3	2	3	1
#71	3	2	3	2
#72	3	2	3	3
#73	3	3	1	1
#74	3	3	1	2
#75	3	3	1	3
#76	3	3	2	1
#77	3	3	2	2
#78	3	3	2	3
#79	3	3	3	1
#80	3	3	3	2
#81	3	3	3	3

Table 9.2.1.1: The 81 possible voting patterns

Step 1

At first, we determine which profile is used by how many voters. Table 9.2.1.2 lists, for every voting pattern, how many voters (column “number of voters”) and which voters (column “voters”) are using this voting pattern.

voting pattern	number of voters	*b*	*c*	*e*	*j*	voters
#1	$w_1^1 = 17$	1	1	1	1	32, 60, 62, 107, 140, 151, 178, 192, 234, 235, 253, 257, 267, 269, 271, 274, 278
#2	$w_1^2 = 2$	1	1	1	2	12, 384
#3	$w_1^3 = 3$	1	1	1	3	255, 258, 270
#4	$w_1^4 = 4$	1	1	2	1	65, 155, 261, 273
#5	$w_1^5 = 4$	1	1	2	2	109, 203, 336, 338
#7	$w_1^7 = 6$	1	1	3	1	59, 90, 166, 249, 348, 367
#9	$w_1^9 = 3$	1	1	3	3	77, 233, 272
#10	$w_1^{10} = 7$	1	2	1	1	11, 174, 189, 195, 239, 242, 260
#11	$w_1^{11} = 7$	1	2	1	2	85, 144, 237, 248, 262, 276, 368
#13	$w_1^{13} = 14$	1	2	2	1	24, 117, 232, 252, 259, 263, 268, 285, 350, 356, 359, 360, 362, 377
#14	$w_1^{14} = 7$	1	2	2	2	93, 105, 214, 221, 240, 339, 448
#19	$w_1^{19} = 18$	1	3	1	1	63, 64, 69, 114, 157, 228, 236, 244, 245, 264, 280, 281, 283, 284, 286, 287, 337, 381
#21	$w_1^{21} = 2$	1	3	1	3	256, 275
#25	$w_1^{25} = 10$	1	3	3	1	73, 112, 193, 216, 229, 251, 266, 282, 361, 373
#27	$w_1^{27} = 17$	1	3	3	3	103, 118, 134, 136, 230, 231, 238, 241, 243, 246, 247, 250, 254, 265, 344, 371, 379
#28	$w_1^{28} = 8$	2	1	1	1	66, 156, 164, 180, 204, 215, 323, 396
#29	$w_1^{29} = 6$	2	1	1	2	16, 67, 162, 332, 395, 397
#31	$w_1^{31} = 2$	2	1	2	1	388, 390
#32	$w_1^{32} = 3$	2	1	2	2	154, 387, 400
#37	$w_1^{37} = 11$	2	2	1	1	15, 18, 115, 133, 209, 210, 222, 324, 326, 358, 392
#38	$w_1^{38} = 7$	2	2	1	2	75, 139, 159, 197, 325, 364, 394
#40	$w_1^{40} = 23$	2	2	2	1	4, 5, 25, 34, 81, 95, 97, 111, 122, 132, 138, 148, 152, 184, 198, 218, 330, 331, 351, 352, 363, 389, 399

Table 9.2.1.2 (1 of 2): voting patterns is example A53

voting pattern	number of voters	b	c	e	j	voters
#41	$w_1^{41} = 13$	2	2	2	2	82, 88, 96, 113, 120, 124, 125, 126, 145, 199, 213, 340, 345
#55	$w_1^{55} = 11$	3	1	1	1	13, 30, 74, 92, 153, 168, 172, 176, 205, 217, 341
#57	$w_1^{57} = 3$	3	1	1	3	14, 376, 380
#61	$w_1^{61} = 10$	3	1	3	1	7, 8, 22, 61, 84, 179, 225, 226, 385, 391
#63	$w_1^{63} = 5$	3	1	3	3	9, 200, 201, 202, 329
#73	$w_1^{73} = 13$	3	3	1	1	23, 31, 70, 76, 79, 188, 190, 194, 207, 208, 277, 378, 386
#75	$w_1^{75} = 14$	3	3	1	3	17, 68, 87, 127, 142, 158, 160, 161, 279, 327, 334, 347, 382, 383
#79	$w_1^{79} = 84$	3	3	3	1	6, 10, 19, 20, 21, 26, 27, 28, 29, 33, 35, 56, 57, 58, 71, 78, 98, 100, 101, 106, 108, 130, 167, 169, 170, 181, 182, 183, 185, 186, 187, 191, 196, 219, 223, 224, 227, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 328, 333, 349, 353, 354, 355, 357, 365, 366, 372, 393, 398
#81	$w_1^{81} = 126$	3	3	3	3	1, 2, 3, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 72, 80, 83, 86, 89, 91, 94, 99, 102, 104, 110, 116, 119, 121, 123, 128, 129, 131, 135, 137, 141, 143, 146, 147, 149, 150, 163, 165, 171, 173, 175, 177, 206, 211, 212, 220, 335, 342, 343, 346, 369, 370, 374, 375, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460

Table 9.2.1.2 (2 of 2): voting patterns is example A53

Step 2

Each time, when we apply proportional completion to a voting pattern, we apply it to a voting pattern, where the number of alternatives with a “2” is the maximum. As, in each stage, a voting pattern is replaced by voting patterns with smaller numbers of alternatives with a “2”, it is guaranteed that those voting patterns, to which proportional completion has already been applied at earlier stages of the proportional completion procedure, cannot reappear at later stages.

So first, we apply proportional completion to voting pattern #41. Applying proportional completion to a voting pattern where voters are indifferent between all candidates simply means that the weight of every other voting pattern is multiplicated by the same factor.

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_2^1 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^1 = 17.494407$	1	1	1	1
#2	$w_2^2 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^2 = 2.058166$	1	1	1	2
#3	$w_2^3 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^3 = 3.087248$	1	1	1	3
#4	$w_2^4 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^4 = 4.116331$	1	1	2	1
#5	$w_2^5 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^5 = 4.116331$	1	1	2	2
#7	$w_2^7 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^7 = 6.174497$	1	1	3	1
#9	$w_2^9 = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^9 = 3.087248$	1	1	3	3
#10	$w_2^{10} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{10} = 7.203579$	1	2	1	1
#11	$w_2^{11} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{11} = 7.203579$	1	2	1	2
#13	$w_2^{13} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{13} = 14.407159$	1	2	2	1
#14	$w_2^{14} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{14} = 7.203579$	1	2	2	2
#19	$w_2^{19} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{19} = 18.523490$	1	3	1	1
#21	$w_2^{21} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{21} = 2.058166$	1	3	1	3
#25	$w_2^{25} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{25} = 10.290828$	1	3	3	1
#27	$w_2^{27} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{27} = 17.494407$	1	3	3	3
#28	$w_2^{28} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{28} = 8.232662$	2	1	1	1
#29	$w_2^{29} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{29} = 6.174497$	2	1	1	2
#31	$w_2^{31} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{31} = 2.058166$	2	1	2	1
#32	$w_2^{32} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{32} = 3.087248$	2	1	2	2
#37	$w_2^{37} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{37} = 11.319911$	2	2	1	1
#38	$w_2^{38} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{38} = 7.203579$	2	2	1	2
#40	$w_2^{40} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{40} = 23.668904$	2	2	2	1
#55	$w_2^{55} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{55} = 11.319911$	3	1	1	1
#57	$w_2^{57} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{57} = 3.087248$	3	1	1	3
#61	$w_2^{61} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{61} = 10.290828$	3	1	3	1
#63	$w_2^{63} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{63} = 5.145414$	3	1	3	3
#73	$w_2^{73} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{73} = 13.378076$	3	3	1	1
#75	$w_2^{75} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{75} = 14.407159$	3	3	1	3
#79	$w_2^{79} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{79} = 86.442953$	3	3	3	1
#81	$w_2^{81} = (1 + w_1^{41} / (N - w_1^{41})) \cdot w_1^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 3

We now apply proportional completion to voting pattern #14. In voting pattern #14, the voters are indifferent between the alternatives in $\{a, c, e, j\}$. At stage 1, $Y := w_1^{14} + w_1^{41} = 20$ voters were indifferent between all the alternatives in $\{a, c, e, j\}$. The following $N - Y = 440$ voters were not indifferent between all the alternatives in $\{a, c, e, j\}$:

number of voters	*c*	*e*	*j*
$w_1^{1} + w_1^{28} + w_1^{55} = 36$	1	1	1
$w_1^{2} + w_1^{29} = 8$	1	1	2
$w_1^{3} + w_1^{57} = 6$	1	1	3
$w_1^{4} + w_1^{31} = 6$	1	2	1
$w_1^{5} + w_1^{32} = 7$	1	2	2
$w_1^{7} + w_1^{61} = 16$	1	3	1
$w_1^{9} + w_1^{63} = 8$	1	3	3
$w_1^{10} + w_1^{37} = 18$	2	1	1
$w_1^{11} + w_1^{38} = 14$	2	1	2
$w_1^{13} + w_1^{40} = 37$	2	2	1
$w_1^{19} + w_1^{73} = 31$	3	1	1
$w_1^{21} + w_1^{75} = 16$	3	1	3
$w_1^{25} + w_1^{79} = 94$	3	3	1
$w_1^{27} + w_1^{81} = 143$	3	3	3
$N - Y = 440$			

Therefore, the w_2^{14} = 7.203579 voters with voting pattern #14 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^{28} + w_1^{55}) \cdot w_2^{14} / (N - Y) = 0.589384$	1	1	1	1
#2	$(w_1^2 + w_1^{29}) \cdot w_2^{14} / (N - Y) = 0.130974$	1	1	1	2
#3	$(w_1^3 + w_1^{57}) \cdot w_2^{14} / (N - Y) = 0.098231$	1	1	1	3
#4	$(w_1^4 + w_1^{31}) \cdot w_2^{14} / (N - Y) = 0.098231$	1	1	2	1
#5	$(w_1^5 + w_1^{32}) \cdot w_2^{14} / (N - Y) = 0.114602$	1	1	2	2
#7	$(w_1^7 + w_1^{61}) \cdot w_2^{14} / (N - Y) = 0.2619483$	1	1	3	1
#9	$(w_1^9 + w_1^{63}) \cdot w_2^{14} / (N - Y) = 0.130974$	1	1	3	3
#10	$(w_1^{10} + w_1^{37}) \cdot w_2^{14} / (N - Y) = 0.294692$	1	2	1	1
#11	$(w_1^{11} + w_1^{38}) \cdot w_2^{14} / (N - Y) = 0.229205$	1	2	1	2
#13	$(w_1^{13} + w_1^{40}) \cdot w_2^{14} / (N - Y) = 0.605756$	1	2	2	1
#19	$(w_1^{19} + w_1^{73}) \cdot w_2^{14} / (N - Y) = 0.507525$	1	3	1	1
#21	$(w_1^{21} + w_1^{75}) \cdot w_2^{14} / (N - Y) = 0.261948$	1	3	1	3
#25	$(w_1^{25} + w_1^{79}) \cdot w_2^{14} / (N - Y) = 1.538947$	1	3	3	1
#27	$(w_1^{27} + w_1^{81}) \cdot w_2^{14} / (N - Y) = 2.341163$	1	3	3	3
	w_2^{14} = 7.203579				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_3^1 = w_2^1 + 0.589384 = 18.083791$	1	1	1	1
#2	$w_3^2 = w_2^2 + 0.130974 = 2.189140$	1	1	1	2
#3	$w_3^3 = w_2^3 + 0.098231 = 3.185479$	1	1	1	3
#4	$w_3^4 = w_2^4 + 0.098231 = 4.214562$	1	1	2	1
#5	$w_3^5 = w_2^5 + 0.114602 = 4.230933$	1	1	2	2
#7	$w_3^7 = w_2^7 + 0.261948 = 6.436445$	1	1	3	1
#9	$w_3^9 = w_2^9 + 0.130974 = 3.218222$	1	1	3	3
#10	$w_3^{10} = w_2^{10} + 0.294692 = 7.498271$	1	2	1	1
#11	$w_3^{11} = w_2^{11} + 0.229205 = 7.432784$	1	2	1	2
#13	$w_3^{13} = w_2^{13} + 0.605756 = 15.012914$	1	2	2	1
#19	$w_3^{19} = w_2^{19} + 0.507525 = 19.031015$	1	3	1	1
#21	$w_3^{21} = w_2^{21} + 0.261948 = 2.320114$	1	3	1	3
#25	$w_3^{25} = w_2^{25} + 1.538947 = 11.829774$	1	3	3	1
#27	$w_3^{27} = w_2^{27} + 2.341163 = 19.835570$	1	3	3	3
#28	$w_3^{28} = w_2^{28} = 8.232662$	2	1	1	1
#29	$w_3^{29} = w_2^{29} = 6.174497$	2	1	1	2
#31	$w_3^{31} = w_2^{31} = 2.058166$	2	1	2	1
#32	$w_3^{32} = w_2^{32} = 3.087248$	2	1	2	2
#37	$w_3^{37} = w_2^{37} = 11.319911$	2	2	1	1
#38	$w_3^{38} = w_2^{38} = 7.203579$	2	2	1	2
#40	$w_3^{40} = w_2^{40} = 23.668904$	2	2	2	1
#55	$w_3^{55} = w_2^{55} = 11.319911$	3	1	1	1
#57	$w_3^{57} = w_2^{57} = 3.087248$	3	1	1	3
#61	$w_3^{61} = w_2^{61} = 10.290828$	3	1	3	1
#63	$w_3^{63} = w_2^{63} = 5.145414$	3	1	3	3
#73	$w_3^{73} = w_2^{73} = 13.378076$	3	3	1	1
#75	$w_3^{75} = w_2^{75} = 14.407159$	3	3	1	3
#79	$w_3^{79} = w_2^{79} = 86.442953$	3	3	3	1
#81	$w_3^{81} = w_2^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 4

We now apply proportional completion to voting pattern #32. In voting pattern #32, the voters are indifferent between the alternatives in $\{a, b, e, j\}$. At stage 1, $Y := w_1^{32} + w_1^{41} = 16$ voters were indifferent between all the alternatives in $\{a, b, e, j\}$. The following $N - Y = 444$ voters were not indifferent between all the alternatives in $\{a, b, e, j\}$:

number of voters	*b*	*e*	*j*
$w_1^1 + w_1^{10} + w_1^{19} = 42$	1	1	1
$w_1^2 + w_1^{11} = 9$	1	1	2
$w_1^3 + w_1^{21} = 5$	1	1	3
$w_1^4 + w_1^{13} = 18$	1	2	1
$w_1^5 + w_1^{14} = 11$	1	2	2
$w_1^7 + w_1^{25} = 16$	1	3	1
$w_1^9 + w_1^{27} = 20$	1	3	3
$w_1^{28} + w_1^{37} = 19$	2	1	1
$w_1^{29} + w_1^{38} = 13$	2	1	2
$w_1^{31} + w_1^{40} = 25$	2	2	1
$w_1^{55} + w_1^{73} = 24$	3	1	1
$w_1^{57} + w_1^{75} = 17$	3	1	3
$w_1^{61} + w_1^{79} = 94$	3	3	1
$w_1^{63} + w_1^{81} = 131$	3	3	3
$N - Y = 444$			

Therefore, the w_3^{32} = 3.087248 voters with voting pattern #32 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^{10} + w_1^{19}) \cdot w_3^{32} / (N - Y) = 0.292037$	1	1	1	1
#2	$(w_1^2 + w_1^{11}) \cdot w_3^{32} / (N - Y) = 0.062579$	1	1	1	2
#3	$(w_1^3 + w_1^{21}) \cdot w_3^{32} / (N - Y) = 0.034766$	1	1	1	3
#4	$(w_1^4 + w_1^{13}) \cdot w_3^{32} / (N - Y) = 0.125159$	1	1	2	1
#5	$(w_1^5 + w_1^{14}) \cdot w_3^{32} / (N - Y) = 0.076486$	1	1	2	2
#7	$(w_1^7 + w_1^{25}) \cdot w_3^{32} / (N - Y) = 0.111252$	1	1	3	1
#9	$(w_1^9 + w_1^{27}) \cdot w_3^{32} / (N - Y) = 0.139065$	1	1	3	3
#28	$(w_1^{28} + w_1^{37}) \cdot w_3^{32} / (N - Y) = 0.132112$	2	1	1	1
#29	$(w_1^{29} + w_1^{38}) \cdot w_3^{32} / (N - Y) = 0.090392$	2	1	1	2
#31	$(w_1^{31} + w_1^{40}) \cdot w_3^{32} / (N - Y) = 0.173832$	2	1	2	1
#55	$(w_1^{55} + w_1^{73}) \cdot w_3^{32} / (N - Y) = 0.166878$	3	1	1	1
#57	$(w_1^{57} + w_1^{75}) \cdot w_3^{32} / (N - Y) = 0.118205$	3	1	1	3
#61	$(w_1^{61} + w_1^{79}) \cdot w_3^{32} / (N - Y) = 0.653607$	3	1	3	1
#63	$(w_1^{63} + w_1^{81}) \cdot w_3^{32} / (N - Y) = 0.910877$	3	1	3	3
	w_3^{32} = 3.087248				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_4^1 = w_3^1 + 0.292037 = 18.375828$	1	1	1	1
#2	$w_4^2 = w_3^2 + 0.062579 = 2.251719$	1	1	1	2
#3	$w_4^3 = w_3^3 + 0.034766 = 3.220245$	1	1	1	3
#4	$w_4^4 = w_3^4 + 0.125159 = 4.339720$	1	1	2	1
#5	$w_4^5 = w_3^5 + 0.076486 = 4.307419$	1	1	2	2
#7	$w_4^7 = w_3^7 + 0.111252 = 6.547697$	1	1	3	1
#9	$w_4^9 = w_3^9 + 0.139065 = 3.357288$	1	1	3	3
#10	$w_4^{10} = w_3^{10} = 7.498271$	1	2	1	1
#11	$w_4^{11} = w_3^{11} = 7.432784$	1	2	1	2
#13	$w_4^{13} = w_3^{13} = 15.012914$	1	2	2	1
#19	$w_4^{19} = w_3^{19} = 19.031015$	1	3	1	1
#21	$w_4^{21} = w_3^{21} = 2.320114$	1	3	1	3
#25	$w_4^{25} = w_3^{25} = 11.829774$	1	3	3	1
#27	$w_4^{27} = w_3^{27} = 19.835570$	1	3	3	3
#28	$w_4^{28} = w_3^{28} + 0.132112 = 8.364774$	2	1	1	1
#29	$w_4^{29} = w_3^{29} + 0.090392 = 6.264889$	2	1	1	2
#31	$w_4^{31} = w_3^{31} + 0.173832 = 2.231997$	2	1	2	1
#37	$w_4^{37} = w_3^{37} = 11.319911$	2	2	1	1
#38	$w_4^{38} = w_3^{38} = 7.203579$	2	2	1	2
#40	$w_4^{40} = w_3^{40} = 23.668904$	2	2	2	1
#55	$w_4^{55} = w_3^{55} + 0.166878 = 11.486789$	3	1	1	1
#57	$w_4^{57} = w_3^{57} + 0.118205 = 3.205454$	3	1	1	3
#61	$w_4^{61} = w_3^{61} + 0.653607 = 10.944434$	3	1	3	1
#63	$w_4^{63} = w_3^{63} + 0.910877 = 6.056291$	3	1	3	3
#73	$w_4^{73} = w_3^{73} = 13.378076$	3	3	1	1
#75	$w_4^{75} = w_3^{75} = 14.407159$	3	3	1	3
#79	$w_4^{79} = w_3^{79} = 86.442953$	3	3	3	1
#81	$w_4^{81} = w_3^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 5

We now apply proportional completion to voting pattern #38. In voting pattern #38, the voters are indifferent between the alternatives in $\{a, b, c, j\}$. At stage 1, $Y := w_1^{38} + w_1^{41} = 20$ voters were indifferent between all the alternatives in $\{a, b, c, j\}$. The following $N - Y = 440$ voters were not indifferent between all the alternatives in $\{a, b, c, j\}$:

number of voters	b	c	j
$w_1^1 + w_1^4 + w_1^7 = 27$	1	1	1
$w_1^2 + w_1^5 = 6$	1	1	2
$w_1^3 + w_1^9 = 6$	1	1	3
$w_1^{10} + w_1^{13} = 21$	1	2	1
$w_1^{11} + w_1^{14} = 14$	1	2	2
$w_1^{19} + w_1^{25} = 28$	1	3	1
$w_1^{21} + w_1^{27} = 19$	1	3	3
$w_1^{28} + w_1^{31} = 10$	2	1	1
$w_1^{29} + w_1^{32} = 9$	2	1	2
$w_1^{37} + w_1^{40} = 34$	2	2	1
$w_1^{55} + w_1^{61} = 21$	3	1	1
$w_1^{57} + w_1^{63} = 8$	3	1	3
$w_1^{73} + w_1^{79} = 97$	3	3	1
$w_1^{75} + w_1^{81} = 140$	3	3	3
$N - Y = 440$			

Therefore, the w_4^{38} = 7.203579 voters with voting pattern #38 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^4 + w_1^7) \cdot w_4^{38} / (N - Y) = 0.442038$	1	1	1	1
#2	$(w_1^2 + w_1^5) \cdot w_4^{38} / (N - Y) = 0.098231$	1	1	1	2
#3	$(w_1^3 + w_1^9) \cdot w_4^{38} / (N - Y) = 0.098231$	1	1	1	3
#10	$(w_1^{10} + w_1^{13}) \cdot w_4^{38} / (N - Y) = 0.343807$	1	2	1	1
#11	$(w_1^{11} + w_1^{14}) \cdot w_4^{38} / (N - Y) = 0.229205$	1	2	1	2
#19	$(w_1^{19} + w_1^{25}) \cdot w_4^{38} / (N - Y) = 0.458410$	1	3	1	1
#21	$(w_1^{21} + w_1^{27}) \cdot w_4^{38} / (N - Y) = 0.311064$	1	3	1	3
#28	$(w_1^{28} + w_1^{31}) \cdot w_4^{38} / (N - Y) = 0.163718$	2	1	1	1
#29	$(w_1^{29} + w_1^{32}) \cdot w_4^{38} / (N - Y) = 0.147346$	2	1	1	2
#37	$(w_1^{37} + w_1^{40}) \cdot w_4^{38} / (N - Y) = 0.556640$	2	2	1	1
#55	$(w_1^{55} + w_1^{61}) \cdot w_4^{38} / (N - Y) = 0.343807$	3	1	1	1
#57	$(w_1^{57} + w_1^{63}) \cdot w_4^{38} / (N - Y) = 0.130974$	3	1	1	3
#73	$(w_1^{73} + w_1^{79}) \cdot w_4^{38} / (N - Y) = 1.588062$	3	3	1	1
#75	$(w_1^{75} + w_1^{81}) \cdot w_4^{38} / (N - Y) = 2.292048$	3	3	1	3
	w_4^{38} = 7.203579				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_5^1 = w_4^1 + 0.442038 = 18.817866$	1	1	1	1
#2	$w_5^2 = w_4^2 + 0.098231 = 2.349950$	1	1	1	2
#3	$w_5^3 = w_4^3 + 0.098231 = 3.318476$	1	1	1	3
#4	$w_5^4 = w_4^4 = 4.339720$	1	1	2	1
#5	$w_5^5 = w_4^5 = 4.307419$	1	1	2	2
#7	$w_5^7 = w_4^7 = 6.547697$	1	1	3	1
#9	$w_5^9 = w_4^9 = 3.357288$	1	1	3	3
#10	$w_5^{10} = w_4^{10} + 0.343807 = 7.842079$	1	2	1	1
#11	$w_5^{11} = w_4^{11} + 0.229205 = 7.661989$	1	2	1	2
#13	$w_5^{13} = w_4^{13} = 15.012914$	1	2	2	1
#19	$w_5^{19} = w_4^{19} + 0.458410 = 19.489424$	1	3	1	1
#21	$w_5^{21} = w_4^{21} + 0.311064 = 2.631178$	1	3	1	3
#25	$w_5^{25} = w_4^{25} = 11.829774$	1	3	3	1
#27	$w_5^{27} = w_4^{27} = 19.835570$	1	3	3	3
#28	$w_5^{28} = w_4^{28} + 0.163718 = 8.528492$	2	1	1	1
#29	$w_5^{29} = w_4^{29} + 0.147346 = 6.412235$	2	1	1	2
#31	$w_5^{31} = w_4^{31} = 2.231997$	2	1	2	1
#37	$w_5^{37} = w_4^{37} + 0.556640 = 11.876551$	2	2	1	1
#40	$w_5^{40} = w_4^{40} = 23.668904$	2	2	2	1
#55	$w_5^{55} = w_4^{55} + 0.343807 = 11.830596$	3	1	1	1
#57	$w_5^{57} = w_4^{57} + 0.130974 = 3.336428$	3	1	1	3
#61	$w_5^{61} = w_4^{61} = 10.944434$	3	1	3	1
#63	$w_5^{63} = w_4^{63} = 6.056291$	3	1	3	3
#73	$w_5^{73} = w_4^{73} + 1.588062 = 14.966138$	3	3	1	1
#75	$w_5^{75} = w_4^{75} + 2.292048 = 16.699207$	3	3	1	3
#79	$w_5^{79} = w_4^{79} = 86.442953$	3	3	3	1
#81	$w_5^{81} = w_4^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 6

We now apply proportional completion to voting pattern #40. In voting pattern #40, the voters are indifferent between the alternatives in {*a*, *b*, *c*, *e*}. At stage 1, $Y := w_1^{40} + w_1^{41} = 36$ voters were indifferent between all the alternatives in {*a*, *b*, *c*, *e*}. The following $N - Y = 424$ voters were not indifferent between all the alternatives in {*a*, *b*, *c*, *e*}:

number of voters	*b*	*c*	*e*
$w_1^1 + w_1^2 + w_1^3 = 22$	1	1	1
$w_1^4 + w_1^5 = 8$	1	1	2
$w_1^7 + w_1^9 = 9$	1	1	3
$w_1^{10} + w_1^{11} = 14$	1	2	1
$w_1^{13} + w_1^{14} = 21$	1	2	2
$w_1^{19} + w_1^{21} = 20$	1	3	1
$w_1^{25} + w_1^{27} = 27$	1	3	3
$w_1^{28} + w_1^{29} = 14$	2	1	1
$w_1^{31} + w_1^{32} = 5$	2	1	2
$w_1^{37} + w_1^{38} = 18$	2	2	1
$w_1^{55} + w_1^{57} = 14$	3	1	1
$w_1^{61} + w_1^{63} = 15$	3	1	3
$w_1^{73} + w_1^{75} = 27$	3	3	1
$w_1^{79} + w_1^{81} = 210$	3	3	3
$N - Y = 424$			

Therefore, the w_5^{40} = 23.668904 voters with voting pattern #40 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^2 + w_1^3) \cdot w_5^{40} / (N - Y) = 1.228103$	1	1	1	1
#4	$(w_1^4 + w_1^5) \cdot w_5^{40} / (N - Y) = 0.446583$	1	1	2	1
#7	$(w_1^7 + w_1^9) \cdot w_5^{40} / (N - Y) = 0.502406$	1	1	3	1
#10	$(w_1^{10} + w_1^{11}) \cdot w_5^{40} / (N - Y) = 0.781520$	1	2	1	1
#13	$(w_1^{13} + w_1^{14}) \cdot w_5^{40} / (N - Y) = 1.172281$	1	2	2	1
#19	$(w_1^{19} + w_1^{21}) \cdot w_5^{40} / (N - Y) = 1.116458$	1	3	1	1
#25	$(w_1^{25} + w_1^{27}) \cdot w_5^{40} / (N - Y) = 1.507218$	1	3	3	1
#28	$(w_1^{28} + w_1^{29}) \cdot w_5^{40} / (N - Y) = 0.781520$	2	1	1	1
#31	$(w_1^{31} + w_1^{32}) \cdot w_5^{40} / (N - Y) = 0.279114$	2	1	2	1
#37	$(w_1^{37} + w_1^{38}) \cdot w_5^{40} / (N - Y) = 1.004812$	2	2	1	1
#55	$(w_1^{55} + w_1^{57}) \cdot w_5^{40} / (N - Y) = 0.781520$	3	1	1	1
#61	$(w_1^{61} + w_1^{63}) \cdot w_5^{40} / (N - Y) = 0.837343$	3	1	3	1
#73	$(w_1^{73} + w_1^{75}) \cdot w_5^{40} / (N - Y) = 1.507218$	3	3	1	1
#79	$(w_1^{79} + w_1^{81}) \cdot w_5^{40} / (N - Y) = 11.722806$	3	3	3	1
	$w_5^{40} = 23.668904$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_6^1 = w_5^1 + 1.228103 = 20.045969$	1	1	1	1
#2	$w_6^2 = w_5^2 = 2.349950$	1	1	1	2
#3	$w_6^3 = w_5^3 = 3.318476$	1	1	1	3
#4	$w_6^4 = w_5^4 + 0.446583 = 4.786304$	1	1	2	1
#5	$w_6^5 = w_5^5 = 4.307419$	1	1	2	2
#7	$w_6^7 = w_5^7 + 0.502406 = 7.050103$	1	1	3	1
#9	$w_6^9 = w_5^9 = 3.357288$	1	1	3	3
#10	$w_6^{10} = w_5^{10} + 0.781520 = 8.623599$	1	2	1	1
#11	$w_6^{11} = w_5^{11} = 7.661989$	1	2	1	2
#13	$w_6^{13} = w_5^{13} + 1.172281 = 16.185195$	1	2	2	1
#19	$w_6^{19} = w_5^{19} + 1.116458 = 20.605882$	1	3	1	1
#21	$w_6^{21} = w_5^{21} = 2.631178$	1	3	1	3
#25	$w_6^{25} = w_5^{25} + 1.507218 = 13.336992$	1	3	3	1
#27	$w_6^{27} = w_5^{27} = 19.835570$	1	3	3	3
#28	$w_6^{28} = w_5^{28} + 0.781520 = 9.310012$	2	1	1	1
#29	$w_6^{29} = w_5^{29} = 6.412235$	2	1	1	2
#31	$w_6^{31} = w_5^{31} + 0.279114 = 2.511112$	2	1	2	1
#37	$w_6^{37} = w_5^{37} + 1.004812 = 12.881363$	2	2	1	1
#55	$w_6^{55} = w_5^{55} + 0.781520 = 12.612116$	3	1	1	1
#57	$w_6^{57} = w_5^{57} = 3.336428$	3	1	1	3
#61	$w_6^{61} = w_5^{61} + 0.837343 = 11.781778$	3	1	3	1
#63	$w_6^{63} = w_5^{63} = 6.056291$	3	1	3	3
#73	$w_6^{73} = w_5^{73} + 1.507218 = 16.473356$	3	3	1	1
#75	$w_6^{75} = w_5^{75} = 16.699207$	3	3	1	3
#79	$w_6^{79} = w_5^{79} + 11.722806 = 98.165759$	3	3	3	1
#81	$w_6^{81} = w_5^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 7

We now apply proportional completion to voting pattern #5. In voting pattern #5, the voters are indifferent between the alternatives in $\{a, e, j\}$. At stage 1, $Y := w_1^5 + w_1^{14} + w_1^{32} + w_1^{41} = 27$ voters were indifferent between all the alternatives in $\{a, e, j\}$. The following $N - Y = 433$ voters were not indifferent between all the alternatives in $\{a, e, j\}$:

number of voters	e	j
$w_1^1 + w_1^{10} + w_1^{19} + w_1^{28} + w_1^{37} + w_1^{55} + w_1^{73} = 85$	1	1
$w_1^2 + w_1^{11} + w_1^{29} + w_1^{38} = 22$	1	2
$w_1^3 + w_1^{21} + w_1^{57} + w_1^{75} = 22$	1	3
$w_1^4 + w_1^{13} + w_1^{31} + w_1^{40} = 43$	2	1
$w_1^7 + w_1^{25} + w_1^{61} + w_1^{79} = 110$	3	1
$w_1^9 + w_1^{27} + w_1^{63} + w_1^{81} = 151$	3	3
$N - Y = 433$		

Therefore, the $w_6^5 = 4.307419$ voters with voting pattern #5 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^1 + w_1^{10} + w_1^{19} + w_1^{28} + w_1^{37} + w_1^{55} + w_1^{73}) \cdot w_6^5 / (N - Y) = 0.845567$	1	1	1	1
#2	$(w_1^2 + w_1^{11} + w_1^{29} + w_1^{38}) \cdot w_6^5 / (N - Y) = 0.218853$	1	1	1	2
#3	$(w_1^3 + w_1^{21} + w_1^{57} + w_1^{75}) \cdot w_6^5 / (N - Y) = 0.218853$	1	1	1	3
#4	$(w_1^4 + w_1^{13} + w_1^{31} + w_1^{40}) \cdot w_6^5 / (N - Y) = 0.427758$	1	1	2	1
#7	$(w_1^7 + w_1^{25} + w_1^{61} + w_1^{79}) \cdot w_6^5 / (N - Y) = 1.094264$	1	1	3	1
#9	$(w_1^9 + w_1^{27} + w_1^{63} + w_1^{81}) \cdot w_6^5 / (N - Y) = 1.502125$	1	1	3	3
	$w_6^5 = 4.307419$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_7^1 = w_6^1 + 0.845567 = 20.891537$	1	1	1	1
#2	$w_7^2 = w_6^2 + 0.218853 = 2.568802$	1	1	1	2
#3	$w_7^3 = w_6^3 + 0.218853 = 3.537329$	1	1	1	3
#4	$w_7^4 = w_6^4 + 0.427758 = 5.214061$	1	1	2	1
#7	$w_7^7 = w_6^7 + 1.094264 = 8.144367$	1	1	3	1
#9	$w_7^9 = w_6^9 + 1.502125 = 4.859413$	1	1	3	3
#10	$w_7^{10} = w_6^{10} = 8.623599$	1	2	1	1
#11	$w_7^{11} = w_6^{11} = 7.661989$	1	2	1	2
#13	$w_7^{13} = w_6^{13} = 16.185195$	1	2	2	1
#19	$w_7^{19} = w_6^{19} = 20.605882$	1	3	1	1
#21	$w_7^{21} = w_6^{21} = 2.631178$	1	3	1	3
#25	$w_7^{25} = w_6^{25} = 13.336992$	1	3	3	1
#27	$w_7^{27} = w_6^{27} = 19.835570$	1	3	3	3
#28	$w_7^{28} = w_6^{28} = 9.310012$	2	1	1	1
#29	$w_7^{29} = w_6^{29} = 6.412235$	2	1	1	2
#31	$w_7^{31} = w_6^{31} = 2.511112$	2	1	2	1
#37	$w_7^{37} = w_6^{37} = 12.881363$	2	2	1	1
#55	$w_7^{55} = w_6^{55} = 12.612116$	3	1	1	1
#57	$w_7^{57} = w_6^{57} = 3.336428$	3	1	1	3
#61	$w_7^{61} = w_6^{61} = 11.781778$	3	1	3	1
#63	$w_7^{63} = w_6^{63} = 6.056291$	3	1	3	3
#73	$w_7^{73} = w_6^{73} = 16.473356$	3	3	1	1
#75	$w_7^{75} = w_6^{75} = 16.699207$	3	3	1	3
#79	$w_7^{79} = w_6^{79} = 98.165759$	3	3	3	1
#81	$w_7^{81} = w_6^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 8

We now apply proportional completion to voting pattern #11. In voting pattern #11, the voters are indifferent between the alternatives in {a, c, j}. At stage 1, $Y := w_1^{11} + w_1^{14} + w_1^{38} + w_1^{41} = 34$ voters were indifferent between all the alternatives in {a, c, j}. The following $N - Y = 426$ voters were not indifferent between all the alternatives in {a, c, j}:

number of voters	c	j
$w_1^1 + w_1^4 + w_1^7 + w_1^{28} + w_1^{31} + w_1^{55} + w_1^{61} = 58$	1	1
$w_1^2 + w_1^5 + w_1^{29} + w_1^{32} = 15$	1	2
$w_1^3 + w_1^9 + w_1^{57} + w_1^{63} = 14$	1	3
$w_1^{10} + w_1^{13} + w_1^{37} + w_1^{40} = 55$	2	1
$w_1^{19} + w_1^{25} + w_1^{73} + w_1^{79} = 125$	3	1
$w_1^{21} + w_1^{27} + w_1^{75} + w_1^{81} = 159$	3	3
$N - Y = 426$		

Therefore, the $w_7^{11} = 7.661989$ voters with voting pattern #11 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^1 + w_1^4 + w_1^7 + w_1^{28} + w_1^{31} + w_1^{55} + w_1^{61}) \cdot w_7^{11} / (N - Y) = 1.043182$	1	1	1	1
#2	$(w_1^2 + w_1^5 + w_1^{29} + w_1^{32}) \cdot w_7^{11} / (N - Y) = 0.269788$	1	1	1	2
#3	$(w_1^3 + w_1^9 + w_1^{57} + w_1^{63}) \cdot w_7^{11} / (N - Y) = 0.251802$	1	1	1	3
#10	$(w_1^{10} + w_1^{13} + w_1^{37} + w_1^{40}) \cdot w_7^{11} / (N - Y) = 0.989224$	1	2	1	1
#19	$(w_1^{19} + w_1^{25} + w_1^{73} + w_1^{79}) \cdot w_7^{11} / (N - Y) = 2.248236$	1	3	1	1
#21	$(w_1^{21} + w_1^{27} + w_1^{75} + w_1^{81}) \cdot w_7^{11} / (N - Y) = 2.859756$	1	3	1	3
	$w_7^{11} = 7.661989$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_8^1 = w_7^1 + 1.043182 = 21.934718$	1	1	1	1
#2	$w_8^2 = w_7^2 + 0.269788 = 2.838591$	1	1	1	2
#3	$w_8^3 = w_7^3 + 0.251802 = 3.789131$	1	1	1	3
#4	$w_8^4 = w_7^4 = 5.214061$	1	1	2	1
#7	$w_8^7 = w_7^7 = 8.144367$	1	1	3	1
#9	$w_8^9 = w_7^9 = 4.859413$	1	1	3	3
#10	$w_8^{10} = w_7^{10} + 0.989224 = 9.612823$	1	2	1	1
#13	$w_8^{13} = w_7^{13} = 16.185199$	1	2	2	1
#19	$w_8^{19} = w_7^{19} + 2.248236 = 22.854118$	1	3	1	1
#21	$w_8^{21} = w_7^{21} + 2.859756 = 5.490934$	1	3	1	3
#25	$w_8^{25} = w_7^{25} = 13.336992$	1	3	3	1
#27	$w_8^{27} = w_7^{27} = 19.835570$	1	3	3	3
#28	$w_8^{28} = w_7^{28} = 9.310012$	2	1	1	1
#29	$w_8^{29} = w_7^{29} = 6.412235$	2	1	1	2
#31	$w_8^{31} = w_7^{31} = 2.511112$	2	1	2	1
#37	$w_8^{37} = w_7^{37} = 12.881363$	2	2	1	1
#55	$w_8^{55} = w_7^{55} = 12.612116$	3	1	1	1
#57	$w_8^{57} = w_7^{57} = 3.336428$	3	1	1	3
#61	$w_8^{61} = w_7^{61} = 11.781778$	3	1	3	1
#63	$w_8^{63} = w_7^{63} = 6.056291$	3	1	3	3
#73	$w_8^{73} = w_7^{73} = 16.473356$	3	3	1	1
#75	$w_8^{75} = w_7^{75} = 16.699207$	3	3	1	3
#79	$w_8^{79} = w_7^{79} = 98.165759$	3	3	3	1
#81	$w_8^{81} = w_7^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 9

We now apply proportional completion to voting pattern #13. In voting pattern #13, the voters are indifferent between the alternatives in {*a*, *c*, *e*}. At stage 1, $Y := w_1^{13} + w_1^{14} + w_1^{40} + w_1^{41} = 57$ voters were indifferent between all the alternatives in {*a*, *c*, *e*}. The following $N - Y = 403$ voters were not indifferent between all the alternatives in {*a*, *c*, *e*}:

number of voters	*c*	*e*
$w_1^1 + w_1^2 + w_1^3 + w_1^{28} + w_1^{29} + w_1^{55} + w_1^{57} = 50$	1	1
$w_1^4 + w_1^5 + w_1^{31} + w_1^{32} = 13$	1	2
$w_1^7 + w_1^9 + w_1^{61} + w_1^{63} = 24$	1	3
$w_1^{10} + w_1^{11} + w_1^{37} + w_1^{38} = 32$	2	1
$w_1^{19} + w_1^{21} + w_1^{73} + w_1^{75} = 47$	3	1
$w_1^{25} + w_1^{27} + w_1^{79} + w_1^{81} = 237$	3	3
$N - Y = 403$		

Therefore, the $w_8^{13} = 16.185195$ voters with voting pattern #13 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^2 + w_1^3 + w_1^{28} + w_1^{29} + w_1^{55} + w_1^{57}) \cdot w_8^{13} / (N - Y) = 2.008089$	1	1	1	1
#4	$(w_1^4 + w_1^5 + w_1^{31} + w_1^{32}) \cdot w_8^{13} / (N - Y) = 0.522103$	1	1	2	1
#7	$(w_1^7 + w_1^9 + w_1^{61} + w_1^{63}) \cdot w_8^{13} / (N - Y) = 0.963883$	1	1	3	1
#10	$(w_1^{10} + w_1^{11} + w_1^{37} + w_1^{38}) \cdot w_8^{13} / (N - Y) = 1.285177$	1	2	1	1
#19	$(w_1^{19} + w_1^{21} + w_1^{73} + w_1^{75}) \cdot w_8^{13} / (N - Y) = 1.887603$	1	3	1	1
#25	$(w_1^{25} + w_1^{27} + w_1^{79} + w_1^{81}) \cdot w_8^{13} / (N - Y) = 9.518340$	1	3	3	1
	$w_8^{13} = 16.185195$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_9^1 = w_8^1 + 2.008089 = 23.942807$	1	1	1	1
#2	$w_9^2 = w_8^2 = 2.838591$	1	1	1	2
#3	$w_9^3 = w_8^3 = 3.789131$	1	1	1	3
#4	$w_9^4 = w_8^4 + 0.522103 = 5.736164$	1	1	2	1
#7	$w_9^7 = w_8^7 + 0.963883 = 9.108249$	1	1	3	1
#9	$w_9^9 = w_8^9 = 4.859413$	1	1	3	3
#10	$w_9^{10} = w_8^{10} + 1.285177 = 10.898000$	1	2	1	1
#19	$w_9^{19} = w_8^{19} + 1.887603 = 24.741722$	1	3	1	1
#21	$w_9^{21} = w_8^{21} = 5.490934$	1	3	1	3
#25	$w_9^{25} = w_8^{25} + 9.518340 = 22.855333$	1	3	3	1
#27	$w_9^{27} = w_8^{27} = 19.835570$	1	3	3	3
#28	$w_9^{28} = w_8^{28} = 9.310012$	2	1	1	1
#29	$w_9^{29} = w_8^{29} = 6.412235$	2	1	1	2
#31	$w_9^{31} = w_8^{31} = 2.511112$	2	1	2	1
#37	$w_9^{37} = w_8^{37} = 12.881363$	2	2	1	1
#55	$w_9^{55} = w_8^{55} = 12.612116$	3	1	1	1
#57	$w_9^{57} = w_8^{57} = 3.336428$	3	1	1	3
#61	$w_9^{61} = w_8^{61} = 11.781778$	3	1	3	1
#63	$w_9^{63} = w_8^{63} = 6.056291$	3	1	3	3
#73	$w_9^{73} = w_8^{73} = 16.473356$	3	3	1	1
#75	$w_9^{75} = w_8^{75} = 16.699207$	3	3	1	3
#79	$w_9^{79} = w_8^{79} = 98.165759$	3	3	3	1
#81	$w_9^{81} = w_8^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 10

We now apply proportional completion to voting pattern #29. In voting pattern #29, the voters are indifferent between the alternatives in {*a*, *b*, *j*}. At stage 1, $Y := w_1^{29} + w_1^{32} + w_1^{38} = 29$ voters were indifferent between all the alternatives in {*a*, *b*, *j*}. The following $N - Y = 431$ voters were not indifferent between all the alternatives in {*a*, *b*, *j*}:

number of voters	*b*	*j*
$w_1^{1} + w_1^{4} + w_1^{7} + w_1^{10} + w_1^{13} + w_1^{19} + w_1^{25} = 76$	1	1
$w_1^{2} + w_1^{5} + w_1^{11} + w_1^{14} = 20$	1	2
$w_1^{3} + w_1^{9} + w_1^{21} + w_1^{27} = 25$	1	3
$w_1^{28} + w_1^{31} + w_1^{37} + w_1^{40} = 44$	2	1
$w_1^{55} + w_1^{61} + w_1^{73} + w_1^{79} = 118$	3	1
$w_1^{57} + w_1^{63} + w_1^{75} + w_1^{81} = 148$	3	3
$N - Y = 431$		

Therefore, the $w_9^{29} = 6.412235$ voters with voting pattern #29 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^{1} + w_1^{4} + w_1^{7} + w_1^{10} + w_1^{13} + w_1^{19} + w_1^{25}) \cdot w_9^{29} / (N - Y) = 1.130696$	1	1	1	1
#2	$(w_1^{2} + w_1^{5} + w_1^{11} + w_1^{14}) \cdot w_9^{29} / (N - Y) = 0.297552$	1	1	1	2
#3	$(w_1^{3} + w_1^{9} + w_1^{21} + w_1^{27}) \cdot w_9^{29} / (N - Y) = 0.371939$	1	1	1	3
#28	$(w_1^{28} + w_1^{31} + w_1^{37} + w_1^{40}) \cdot w_9^{29} / (N - Y) = 0.654613$	2	1	1	1
#55	$(w_1^{55} + w_1^{61} + w_1^{73} + w_1^{79}) \cdot w_9^{29} / (N - Y) = 1.755554$	3	1	1	1
#57	$(w_1^{57} + w_1^{63} + w_1^{75} + w_1^{81}) \cdot w_9^{29} / (N - Y) = 2.201881$	3	1	1	3
	$w_9^{29} = 6.412235$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{10}^{1} = w_{9}^{1} + 1.130696 = 25.073503$	1	1	1	1
#2	$w_{10}^{2} = w_{9}^{2} + 0.297552 = 3.136142$	1	1	1	2
#3	$w_{10}^{3} = w_{9}^{3} + 0.371939 = 4.161070$	1	1	1	3
#4	$w_{10}^{4} = w_{9}^{4} = 5.736164$	1	1	2	1
#7	$w_{10}^{7} = w_{9}^{7} = 9.108249$	1	1	3	1
#9	$w_{10}^{9} = w_{9}^{9} = 4.859413$	1	1	3	3
#10	$w_{10}^{10} = w_{9}^{10} = 10.898000$	1	2	1	1
#19	$w_{10}^{19} = w_{9}^{19} = 24.741722$	1	3	1	1
#21	$w_{10}^{21} = w_{9}^{21} = 5.490934$	1	3	1	3
#25	$w_{10}^{25} = w_{9}^{25} = 22.855333$	1	3	3	1
#27	$w_{10}^{27} = w_{9}^{27} = 19.835570$	1	3	3	3
#28	$w_{10}^{28} = w_{9}^{28} + 0.654613 = 9.964626$	2	1	1	1
#31	$w_{10}^{31} = w_{9}^{31} = 2.511112$	2	1	2	1
#37	$w_{10}^{37} = w_{9}^{37} = 12.881363$	2	2	1	1
#55	$w_{10}^{55} = w_{9}^{55} + 1.755554 = 14.367670$	3	1	1	1
#57	$w_{10}^{57} = w_{9}^{57} + 2.201881 = 5.538309$	3	1	1	3
#61	$w_{10}^{61} = w_{9}^{61} = 11.781778$	3	1	3	1
#63	$w_{10}^{63} = w_{9}^{63} = 6.056291$	3	1	3	3
#73	$w_{10}^{73} = w_{9}^{73} = 16.473356$	3	3	1	1
#75	$w_{10}^{75} = w_{9}^{75} = 16.699207$	3	3	1	3
#79	$w_{10}^{79} = w_{9}^{79} = 98.165759$	3	3	3	1
#81	$w_{10}^{81} = w_{9}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 11

We now apply proportional completion to voting pattern #31. In voting pattern #31, the voters are indifferent between the alternatives in {*a*, *b*, *e*}. At stage 1, $Y := w_1^{31} + w_1^{32} + w_1^{40} + w_1^{41} = 41$ voters were indifferent between all the alternatives in {*a*, *b*, *e*}. The following $N - Y = 419$ voters were not indifferent between all the alternatives in {*a*, *b*, *e*}:

number of voters	*b*	*e*
$w_1^1 + w_1^2 + w_1^3 + w_1^{10} + w_1^{11} + w_1^{19} + w_1^{21} = 56$	1	1
$w_1^4 + w_1^5 + w_1^{13} + w_1^{14} = 29$	1	2
$w_1^7 + w_1^9 + w_1^{25} + w_1^{27} = 36$	1	3
$w_1^{28} + w_1^{29} + w_1^{37} + w_1^{38} = 32$	2	1
$w_1^{55} + w_1^{57} + w_1^{73} + w_1^{75} = 41$	3	1
$w_1^{61} + w_1^{63} + w_1^{79} + w_1^{81} = 225$	3	3
$N - Y = 419$		

Therefore, the $w_{10}^{31} = 2.511112$ voters with voting pattern #31 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^2 + w_1^3 + w_1^{10} + w_1^{11} + w_1^{19} + w_1^{21}) \cdot w_{10}^{31} / (N - Y) = 0.335614$	1	1	1	1
#4	$(w_1^4 + w_1^5 + w_1^{13} + w_1^{14}) \cdot w_{10}^{31} / (N - Y) = 0.173800$	1	1	2	1
#7	$(w_1^7 + w_1^9 + w_1^{25} + w_1^{27}) \cdot w_{10}^{31} / (N - Y) = 0.215752$	1	1	3	1
#28	$(w_1^{28} + w_1^{29} + w_1^{37} + w_1^{38}) \cdot w_{10}^{31} / (N - Y) = 0.191779$	2	1	1	1
#55	$(w_1^{55} + w_1^{57} + w_1^{73} + w_1^{75}) \cdot w_{10}^{31} / (N - Y) = 0.245717$	3	1	1	1
#61	$(w_1^{61} + w_1^{63} + w_1^{79} + w_1^{81}) \cdot w_{10}^{31} / (N - Y) = 1.348449$	3	1	3	1
	$w_{10}^{31} = 2.511112$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{11}^{1} = w_{10}^{1} + 0.335614 = 25.409117$	1	1	1	1
#2	$w_{11}^{2} = w_{10}^{2} = 3.136142$	1	1	1	2
#3	$w_{11}^{3} = w_{10}^{3} = 4.161070$	1	1	1	3
#4	$w_{11}^{4} = w_{10}^{4} + 0.173800 = 5.909964$	1	1	2	1
#7	$w_{11}^{7} = w_{10}^{7} + 0.215752 = 9.324001$	1	1	3	1
#9	$w_{11}^{9} = w_{10}^{9} = 4.859413$	1	1	3	3
#10	$w_{11}^{10} = w_{10}^{10} = 10.898000$	1	2	1	1
#19	$w_{11}^{19} = w_{10}^{19} = 24.741722$	1	3	1	1
#21	$w_{11}^{21} = w_{10}^{21} = 5.490934$	1	3	1	3
#25	$w_{11}^{25} = w_{10}^{25} = 22.855333$	1	3	3	1
#27	$w_{11}^{27} = w_{10}^{27} = 19.835570$	1	3	3	3
#28	$w_{11}^{28} = w_{10}^{28} + 0.191779 = 10.156405$	2	1	1	1
#37	$w_{11}^{37} = w_{10}^{37} = 12.881363$	2	2	1	1
#55	$w_{11}^{55} = w_{10}^{55} + 0.245717 = 14.613388$	3	1	1	1
#57	$w_{11}^{57} = w_{10}^{57} = 5.538309$	3	1	1	3
#61	$w_{11}^{61} = w_{10}^{61} + 1.348449 = 13.130227$	3	1	3	1
#63	$w_{11}^{63} = w_{10}^{63} = 6.056291$	3	1	3	3
#73	$w_{11}^{73} = w_{10}^{73} = 16.473356$	3	3	1	1
#75	$w_{11}^{75} = w_{10}^{75} = 16.699207$	3	3	1	3
#79	$w_{11}^{79} = w_{10}^{79} = 98.165759$	3	3	3	1
#81	$w_{11}^{81} = w_{10}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 12

We now apply proportional completion to voting pattern #37. In voting pattern #37, the voters are indifferent between the alternatives in {*a*, *b*, *c*}. At stage 1, $Y := w_1^{37} + w_1^{38} + w_1^{40} + w_1^{41} = 54$ voters were indifferent between all the alternatives in {*a*, *b*, *c*}. The following $N - Y = 406$ voters were not indifferent between all the alternatives in {*a*, *b*, *c*}:

number of voters	*b*	*c*
$w_1^1 + w_1^2 + w_1^3 + w_1^4 + w_1^5 + w_1^7 + w_1^9 = 39$	1	1
$w_1^{10} + w_1^{11} + w_1^{13} + w_1^{14} = 35$	1	2
$w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27} = 47$	1	3
$w_1^{28} + w_1^{29} + w_1^{31} + w_1^{32} = 19$	2	1
$w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63} = 29$	3	1
$w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81} = 237$	3	3
$N - Y = 406$		

Therefore, the $w_{11}^{37} = 12.881363$ voters with voting pattern #37 are replaced by the following voters:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$(w_1^1 + w_1^2 + w_1^3 + w_1^4 + w_1^5 + w_1^7 + w_1^9) \cdot w_{11}^{37} / (N - Y) = 1.237372$	1	1	1	1
#10	$(w_1^{10} + w_1^{11} + w_1^{13} + w_1^{14}) \cdot w_{11}^{37} / (N - Y) = 1.110462$	1	2	1	1
#19	$(w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27}) \cdot w_{11}^{37} / (N - Y) = 1.491192$	1	3	1	1
#28	$(w_1^{28} + w_1^{29} + w_1^{31} + w_1^{32}) \cdot w_{11}^{37} / (N - Y) = 0.602822$	2	1	1	1
#55	$(w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63}) \cdot w_{11}^{37} / (N - Y) = 0.920097$	3	1	1	1
#73	$(w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81}) \cdot w_{11}^{37} / (N - Y) = 7.519416$	3	3	1	1
	$w_{11}^{37} = 12.881363$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{12}^{1} = w_{11}^{1} + 1.237372 = 26.646489$	1	1	1	1
#2	$w_{12}^{2} = w_{11}^{2} = 3.136142$	1	1	1	2
#3	$w_{12}^{3} = w_{11}^{3} = 4.161070$	1	1	1	3
#4	$w_{12}^{4} = w_{11}^{4} = 5.909964$	1	1	2	1
#7	$w_{12}^{7} = w_{11}^{7} = 9.324001$	1	1	3	1
#9	$w_{12}^{9} = w_{11}^{9} = 4.859413$	1	1	3	3
#10	$w_{12}^{10} = w_{11}^{10} + 1.110462 = 12.008462$	1	2	1	1
#19	$w_{12}^{19} = w_{11}^{19} + 1.491192 = 26.232914$	1	3	1	1
#21	$w_{12}^{21} = w_{11}^{21} = 5.490934$	1	3	1	3
#25	$w_{12}^{25} = w_{11}^{25} = 22.855333$	1	3	3	1
#27	$w_{12}^{27} = w_{11}^{27} = 19.835570$	1	3	3	3
#28	$w_{12}^{28} = w_{11}^{28} + 0.602822 = 10.759227$	2	1	1	1
#55	$w_{12}^{55} = w_{11}^{55} + 0.920097 = 15.533485$	3	1	1	1
#57	$w_{12}^{57} = w_{11}^{57} = 5.538309$	3	1	1	3
#61	$w_{12}^{61} = w_{11}^{61} = 13.130227$	3	1	3	1
#63	$w_{12}^{63} = w_{11}^{63} = 6.056291$	3	1	3	3
#73	$w_{12}^{73} = w_{11}^{73} + 7.519416 = 23.992772$	3	3	1	1
#75	$w_{12}^{75} = w_{11}^{75} = 16.699207$	3	3	1	3
#79	$w_{12}^{79} = w_{11}^{79} = 98.165759$	3	3	3	1
#81	$w_{12}^{81} = w_{11}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 13

We now apply proportional completion to voting pattern #2. In voting pattern #2, the voters are indifferent between the alternatives in $\{a, j\}$. At stage 1, $Y := w_1^2 + w_1^5 + w_1^{11} + w_1^{14} + w_1^{29} + w_1^{32} + w_1^{38} + w_1^{41} = 49$ voters were indifferent between all the alternatives in $\{a, j\}$. The following $N - Y$ = 411 voters were not indifferent between all the alternatives in $\{a, j\}$:

number of voters	j
$w_1^1 + w_1^4 + w_1^7 + w_1^{10} + w_1^{13} + w_1^{19} + w_1^{25} + w_1^{28} + w_1^{31} + w_1^{37} + w_1^{40} + w_1^{55} + w_1^{61} + w_1^{73} + w_1^{79} = 238$	1
$w_1^3 + w_1^9 + w_1^{21} + w_1^{27} + w_1^{57} + w_1^{63} + w_1^{75} + w_1^{81} = 173$	3
$N - Y = 411$	

Therefore, the $w_{12}^2 = 3.136142$ voters with voting pattern #2 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^1 + w_1^4 + w_1^7 + w_1^{10} + w_1^{13} + w_1^{19} + w_1^{25} + w_1^{28} + w_1^{31} + w_1^{37} + w_1^{40} + w_1^{55} + w_1^{61} + w_1^{73} + w_1^{79}) \cdot w_{12}^2 / (N - Y) = 1.816063$	1	1	1	1
#3	$(w_1^3 + w_1^9 + w_1^{21} + w_1^{27} + w_1^{57} + w_1^{63} + w_1^{75} + w_1^{81}) \cdot w_{12}^2 / (N - Y) = 1.320079$	1	1	1	3
	$w_{12}^2 = 3.136142$				

Therefore, we get:

voting pattern	number of voters	b	c	e	j
#1	$w_{13}^{1} = w_{12}^{1} + 1.816063 = 28.462552$	1	1	1	1
#3	$w_{13}^{3} = w_{12}^{3} + 1.320079 = 5.481150$	1	1	1	3
#4	$w_{13}^{4} = w_{12}^{4} = 5.909964$	1	1	2	1
#7	$w_{13}^{7} = w_{12}^{7} = 9.324001$	1	1	3	1
#9	$w_{13}^{9} = w_{12}^{9} = 4.859413$	1	1	3	3
#10	$w_{13}^{10} = w_{12}^{10} = 12.008462$	1	2	1	1
#19	$w_{13}^{19} = w_{12}^{19} = 26.232914$	1	3	1	1
#21	$w_{13}^{21} = w_{12}^{21} = 5.490934$	1	3	1	3
#25	$w_{13}^{25} = w_{12}^{25} = 22.855333$	1	3	3	1
#27	$w_{13}^{27} = w_{12}^{27} = 19.835570$	1	3	3	3
#28	$w_{13}^{28} = w_{12}^{28} = 10.759227$	2	1	1	1
#55	$w_{13}^{55} = w_{12}^{55} = 15.533485$	3	1	1	1
#57	$w_{13}^{57} = w_{12}^{57} = 5.538309$	3	1	1	3
#61	$w_{13}^{61} = w_{12}^{61} = 13.130227$	3	1	3	1
#63	$w_{13}^{63} = w_{12}^{63} = 6.056291$	3	1	3	3
#73	$w_{13}^{73} = w_{12}^{73} = 23.992772$	3	3	1	1
#75	$w_{13}^{75} = w_{12}^{75} = 16.699207$	3	3	1	3
#79	$w_{13}^{79} = w_{12}^{79} = 98.165759$	3	3	3	1
#81	$w_{13}^{81} = w_{12}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 14

We now apply proportional completion to voting pattern #4. In voting pattern #4, the voters are indifferent between the alternatives in $\{a, e\}$. At stage 1, $Y := w_1^4 + w_1^5 + w_1^{13} + w_1^{14} + w_1^{31} + w_1^{32} + w_1^{40} + w_1^{41} = 70$ voters were indifferent between all the alternatives in $\{a, e\}$. The following $N - Y$ = 390 voters were not indifferent between all the alternatives in $\{a, e\}$:

number of voters	e
$w_1^1 + w_1^2 + w_1^3 + w_1^{10} + w_1^{11} + w_1^{19} + w_1^{21} + w_1^{28} + w_1^{29} + w_1^{37} + w_1^{38} + w_1^{55} + w_1^{57} + w_1^{73} + w_1^{75} = 129$	1
$w_1^7 + w_1^9 + w_1^{25} + w_1^{27} + w_1^{61} + w_1^{63} + w_1^{79} + w_1^{81} = 261$	3
$N - Y = 390$	

Therefore, the $w_{13}^4 = 5.909964$ voters with voting pattern #4 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^1 + w_1^2 + w_1^3 + w_1^{10} + w_1^{11} + w_1^{19} + w_1^{21} + w_1^{28} + w_1^{29} + w_1^{37} + w_1^{38} + w_1^{55} + w_1^{57} + w_1^{73} + w_1^{75}) \cdot w_{13}^4 / (N - Y) = 1.954834$	1	1	1	1
#7	$(w_1^7 + w_1^9 + w_1^{25} + w_1^{27} + w_1^{61} + w_1^{63} + w_1^{79} + w_1^{81}) \cdot w_{13}^4 / (N - Y) = 3.955130$	1	1	3	1
	$w_{13}^4 = 5.909964$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{14}^{1} = w_{13}^{1} + 1.954834 = 30.417386$	1	1	1	1
#3	$w_{14}^{3} = w_{13}^{3} = 5.481150$	1	1	1	3
#7	$w_{14}^{7} = w_{13}^{7} + 3.955130 = 13.279131$	1	1	3	1
#9	$w_{14}^{9} = w_{13}^{9} = 4.859413$	1	1	3	3
#10	$w_{14}^{10} = w_{13}^{10} = 12.008462$	1	2	1	1
#19	$w_{14}^{19} = w_{13}^{19} = 26.232914$	1	3	1	1
#21	$w_{14}^{21} = w_{13}^{21} = 5.490934$	1	3	1	3
#25	$w_{14}^{25} = w_{13}^{25} = 22.855333$	1	3	3	1
#27	$w_{14}^{27} = w_{13}^{27} = 19.835570$	1	3	3	3
#28	$w_{14}^{28} = w_{13}^{28} = 10.759227$	2	1	1	1
#55	$w_{14}^{55} = w_{13}^{55} = 15.533485$	3	1	1	1
#57	$w_{14}^{57} = w_{13}^{57} = 5.538309$	3	1	1	3
#61	$w_{14}^{61} = w_{13}^{61} = 13.130227$	3	1	3	1
#63	$w_{14}^{63} = w_{13}^{63} = 6.056291$	3	1	3	3
#73	$w_{14}^{73} = w_{13}^{73} = 23.992772$	3	3	1	1
#75	$w_{14}^{75} = w_{13}^{75} = 16.699207$	3	3	1	3
#79	$w_{14}^{79} = w_{13}^{79} = 98.165759$	3	3	3	1
#81	$w_{14}^{81} = w_{13}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 15

We now apply proportional completion to voting pattern #10. In voting pattern #10, the voters are indifferent between the alternatives in $\{a, c\}$. At stage 1, $Y := w_1^{10} + w_1^{11} + w_1^{13} + w_1^{14} + w_1^{37} + w_1^{38} + w_1^{40} + w_1^{41} = 89$ voters were indifferent between all the alternatives in $\{a, c\}$. The following $N - Y = 371$ voters were not indifferent between all the alternatives in $\{a, c\}$:

number of voters	c
$w_1^{1} + w_1^{2} + w_1^{3} + w_1^{4} + w_1^{5} + w_1^{7} + w_1^{9} + w_1^{28} + w_1^{29} + w_1^{31} + w_1^{32} + w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63} = 87$	1
$w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27} + w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81} = 284$	3
$N - Y = 371$	

Therefore, the $w_{14}^{10} = 12.008462$ voters with voting pattern #10 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^{1} + w_1^{2} + w_1^{3} + w_1^{4} + w_1^{5} + w_1^{7} + w_1^{9} + w_1^{28} + w_1^{29} + w_1^{31} + w_1^{32} + w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63}) \cdot w_{14}^{10} / (N - Y) = 2.816001$	1	1	1	1
#19	$(w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27} + w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81}) \cdot w_{14}^{10} / (N - Y) = 9.192461$	1	3	1	1
	$w_{14}^{10} = 12.008462$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{15}^{1} = w_{14}^{1} + 2.816001 = 33.233387$	1	1	1	1
#3	$w_{15}^{3} = w_{14}^{3} = 5.481150$	1	1	1	3
#7	$w_{15}^{7} = w_{14}^{7} = 13.279131$	1	1	3	1
#9	$w_{15}^{9} = w_{14}^{9} = 4.859413$	1	1	3	3
#19	$w_{15}^{19} = w_{14}^{19} + 9.192461 = 35.425375$	1	3	1	1
#21	$w_{15}^{21} = w_{14}^{21} = 5.490934$	1	3	1	3
#25	$w_{15}^{25} = w_{14}^{25} = 22.855333$	1	3	3	1
#27	$w_{15}^{27} = w_{14}^{27} = 19.835570$	1	3	3	3
#28	$w_{15}^{28} = w_{14}^{28} = 10.759227$	2	1	1	1
#55	$w_{15}^{55} = w_{14}^{55} = 15.533485$	3	1	1	1
#57	$w_{15}^{57} = w_{14}^{57} = 5.538309$	3	1	1	3
#61	$w_{15}^{61} = w_{14}^{61} = 13.130227$	3	1	3	1
#63	$w_{15}^{63} = w_{14}^{63} = 6.056291$	3	1	3	3
#73	$w_{15}^{73} = w_{14}^{73} = 23.992772$	3	3	1	1
#75	$w_{15}^{75} = w_{14}^{75} = 16.699207$	3	3	1	3
#79	$w_{15}^{79} = w_{14}^{79} = 98.165759$	3	3	3	1
#81	$w_{15}^{81} = w_{14}^{81} = 129.664430$	3	3	3	3
	460.000000				

Step 16

We now apply proportional completion to voting pattern #28. In voting pattern #28, the voters are indifferent between the alternatives in {a, b}. At stage 1, $Y := w_1^{28} + w_1^{29} + w_1^{31} + w_1^{32} + w_1^{37} + w_1^{38} + w_1^{40} + w_1^{41} = 73$ voters were indifferent between all the alternatives in {a, b}. The following $N - Y =$ 387 voters were not indifferent between all the alternatives in {a, b}:

number of voters	b
$w_1^1 + w_1^2 + w_1^3 + w_1^4 + w_1^5 + w_1^7 + w_1^9 + w_1^{10} + w_1^{11} + w_1^{13} + w_1^{14} + w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27} = 121$	1
$w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63} + w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81} = 266$	3
$N - Y = 387$	

Therefore, the $w_{15}^{28} = 10.759227$ voters with voting pattern #28 are replaced by the following voters:

voting pattern	number of voters	b	c	e	j
#1	$(w_1^1 + w_1^2 + w_1^3 + w_1^4 + w_1^5 + w_1^7 + w_1^9 + w_1^{10} + w_1^{11} + w_1^{13} + w_1^{14} + w_1^{19} + w_1^{21} + w_1^{25} + w_1^{27}) \cdot w_{15}^{28} / (N - Y) = 3.363996$	1	1	1	1
#55	$(w_1^{55} + w_1^{57} + w_1^{61} + w_1^{63} + w_1^{73} + w_1^{75} + w_1^{79} + w_1^{81}) \cdot w_{15}^{28} / (N - Y) = 7.395231$	3	1	1	1
	$w_{15}^{28} = 10.759227$				

Therefore, we get:

voting pattern	number of voters	*b*	*c*	*e*	*j*
#1	$w_{16}^{1} = w_{15}^{1} + 3.363996 = 36.597383$	1	1	1	1
#3	$w_{16}^{3} = w_{15}^{3} = 5.481150$	1	1	1	3
#7	$w_{16}^{7} = w_{15}^{7} = 13.279131$	1	1	3	1
#9	$w_{16}^{9} = w_{15}^{9} = 4.859413$	1	1	3	3
#19	$w_{16}^{19} = w_{15}^{19} = 35.425375$	1	3	1	1
#21	$w_{16}^{21} = w_{15}^{21} = 5.490934$	1	3	1	3
#25	$w_{16}^{25} = w_{15}^{25} = 22.855333$	1	3	3	1
#27	$w_{16}^{27} = w_{15}^{27} = 19.835570$	1	3	3	3
#55	$w_{16}^{55} = w_{15}^{55} + 7.395231 = 22.928716$	3	1	1	1
#57	$w_{16}^{57} = w_{15}^{57} = 5.538309$	3	1	1	3
#61	$w_{16}^{61} = w_{15}^{61} = 13.130227$	3	1	3	1
#63	$w_{16}^{63} = w_{15}^{63} = 6.056291$	3	1	3	3
#73	$w_{16}^{73} = w_{15}^{73} = 23.992772$	3	3	1	1
#75	$w_{16}^{75} = w_{15}^{75} = 16.699207$	3	3	1	3
#79	$w_{16}^{79} = w_{15}^{79} = 98.165759$	3	3	3	1
#81	$w_{16}^{81} = w_{15}^{81} = 129.664430$	3	3	3	3
	460.000000				

9.2.2. Links between Sets of Winners

In section 9.2.2, we show how the strengths of the links are calculated.

$N[\{a_1,...,a_M\};b]$ is the strength of the link from the set $\{a_1,...,a_M\}$ to the set $\{a_1,...,a_{(M-1)},b\}$. $N[\{a_1,...,a_M\};b]$ is defined as follows:

$N[\{a_1,...,a_M\};b] \in \mathbb{R}$ is the largest value such that there is a $t \in \mathbb{R}^{(N_W \times M)}$ such that:

(9.1.2.1) $\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $t_{ij} \geq 0$.

(9.1.2.2) $\forall\, i \in \{1,...,N_W\}$: $\sum_{j=1}^{M} t_{ij} \leq \rho(i)$.

(9.1.2.3) $\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $b \succ_i a_j \Rightarrow t_{ij} = 0$.

(9.1.2.4) $\forall\, j \in \{1,...,M\}$: $\sum_{i=1}^{N_W} t_{ij} \geq N[\{a_1,...,a_M\};b]$.

Suppose $N^*[\{a_1,...,a_M\};b] \in \mathbb{R}$ is the largest value such that there is a $t^* \in \mathbb{R}^{(N_W \times M)}$ such that:

(9.2.2.1) $\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $t^*_{ij} \geq 0$.

(9.2.2.2) $\forall\, i \in \{1,...,N_W\}$: $\sum_{j=1}^{M} t^*_{ij} \leq \rho(i)$.

(9.2.2.3) $\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}$: $b \succ_i a_j \Rightarrow t^*_{ij} = 0$.

(9.2.2.4) $\sum_{i=1}^{N_W} \sum_{j=1}^{M} t^*_{ij} \geq M \cdot N^*[\{a_1,...,a_M\};b]$.

As (9.2.2.4) is weaker than (9.1.2.4), we get:

$$N[\{a_1,...,a_M\};b] \leq N^*[\{a_1,...,a_M\};b].$$

Suppose $t^* \in \mathbb{R}^{(N_W \times M)}$ is a solution of (9.2.2.1) – (9.2.2.4). Then we define:

$$N\hat{\ }[\{a_1,...,a_M\};b] := \min \{ \sum_{i=1}^{N_W} t^*_{ij} \mid 1 \leq j \leq M \}.$$

So we get:

$$N\hat{\ }[\{a_1,...,a_M\};b] \leq N[\{a_1,...,a_M\};b] \leq N^*[\{a_1,...,a_M\};b].$$

Compared to (9.1.2.1) – (9.1.2.4), (9.2.2.1) – (9.2.2.4) has the advantage that it describes a trivial max-flow problem. A max-flow problem can be solved significantly more easily than a general linear program. Therefore, we solve (9.1.2.1) – (9.1.2.4) by solving a series of max-flow problems as follows:

Suppose $\mathbf{W}$ is the number of voters who strictly prefer candidate b to every candidate of the set $\{a_1,...,a_M\}$. Then we know that $N[\{a_1,...,a_M\};b]$ cannot be larger than $(N - \mathbf{W}) / M$.

Therefore, we start with

$$r^{(0)} := (N - \mathbf{W}) / M.$$

$$s^{(0)} := 0.$$

For $z = 1, 2, 3, ...$, we solve the following linear programs $\mathsf{LP}^{(z)}$:

Find the maximum $r^{(z)} \in \mathbb{R}$ such that there is a $t^{(z)} \in \mathbb{R}^{(N_W \times M)}$ such that

(9.2.2.5) $$\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}\colon t_{ij}^{(z)} \geq 0.$$

(9.2.2.6) $$\forall\, i \in \{1,...,N_W\}\colon \sum_{j=1}^{M} t_{ij}^{(z)} \leq \rho(i).$$

(9.2.2.7) $$\forall\, i \in \{1,...,N_W\}\ \forall\, j \in \{1,...,M\}\colon b \succ_i a_j \Rightarrow t_{ij}^{(z)} = 0.$$

(9.2.2.8) $$\sum_{i=1}^{N_W} \sum_{j=1}^{M} t_{ij}^{(z)} \geq M \cdot r^{(z)}.$$

(9.2.2.9) $$\forall\, j \in \{1,...,M\}\colon \sum_{i=1}^{N_W} t_{ij}^{(z)} \leq r^{(z-1)}.$$

Furthermore, we define for $z = 1, 2, 3, ...$:

(9.2.2.10) $$s^{(z)} := \max \left\{ s^{(z-1)}, \min \left\{ \sum_{i=1}^{N_W} t_{ij}^{(z)} \ \middle|\ 1 \leq j \leq M \right\} \right\}.$$

When we solve (9.2.2.5) – (9.2.2.9), then we get a decreasing sequence $r^{(0)}$, $r^{(1)}$, $r^{(2)}$, $r^{(3)}$, ... and an increasing sequence $s^{(0)}$, $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, ... These two sequences converge to the same limit. This limit is the solution of (9.1.2.1) – (9.1.2.4).

Now, we use this algorithm to calculate the strength of the link of the alternatives *b*,*c*,*e*,*j* against candidate *a* in instance A53. After proportional completion, the voter profile looks as follows:

		b	*c*	*e*	*j*
voter01	36.597383	1	1	1	1
voter02	5.481150	1	1	1	3
voter03	13.279131	1	1	3	1
voter04	4.859413	1	1	3	3
voter05	35.425375	1	3	1	1
voter06	5.490934	1	3	1	3
voter07	22.855333	1	3	3	1
voter08	19.835570	1	3	3	3
voter09	22.928716	3	1	1	1
voter10	5.538309	3	1	1	3
voter11	13.130227	3	1	3	1
voter12	6.056291	3	1	3	3
voter13	23.992772	3	3	1	1
voter14	16.699207	3	3	1	3
voter15	98.165759	3	3	3	1
voter16	129.664430	3	3	3	3
	460.000000				

The corresponding max-flow problem has the following form:

> Each voting pattern, where voters strictly prefer at least one alternative of the set {*b*,*c*,*e*,*j*} to alternative *a*, is represented by a vertex. Each alternative of the set {*b*,*c*,*e*,*j*} is represented by a vertex. Furthermore, there is a vertex “source” and a vertex “drain”.
>
> From the vertex “source” we draw a link to each vertex that represents a voting pattern. The maximum capacity of this link is the number of voters with this voting pattern.
>
> From each vertex, that represents a voting pattern, we draw a link to each vertex that represents an alternative that is strictly preferred to alternative *a* by voters with this voting pattern. The maximum capacity of this link is the number of voters with this voting pattern.
>
> From each vertex, that represents an alternative, we draw a link to the vertex “drain”. The maximum capacity of this link is $r^{(z-1)}$.
>
> The task is: Maximize the total flow from the vertex “source” to the vertex “drain”.

In our case, we get a digraph with 21 vertices and 51 links.

Furthermore, we get:

$$r^{(0)} : = (N - \varpi) / M = (460 - 129.664430) / 4 = 82.583893$$

Our digraph has the following form:

link	start	end	capacity
1	source	voter01	36.597383
2	source	voter02	5.481150
3	source	voter03	13.279131
4	source	voter04	4.859413
5	source	voter05	35.425375
6	source	voter06	5.490934
7	source	voter07	22.855333
8	source	voter08	19.835570
9	source	voter09	22.928716
10	source	voter10	5.538309
11	source	voter11	13.130227
12	source	voter12	6.056291
13	source	voter13	23.992772
14	source	voter14	16.699207
15	source	voter15	98.165759
16	voter01	alternative *b*	36.597383
17	voter01	alternative *c*	36.597383
18	voter01	alternative *e*	36.597383
19	voter01	alternative *j*	36.597383
20	voter02	alternative *b*	5.481150
21	voter02	alternative *c*	5.481150
22	voter02	alternative *e*	5.481150
23	voter03	alternative *b*	13.279131
24	voter03	alternative *c*	13.279131
25	voter03	alternative *j*	13.279131
26	voter04	alternative *b*	4.859413
27	voter04	alternative *c*	4.859413
28	voter05	alternative *b*	35.425375
29	voter05	alternative *e*	35.425375
30	voter05	alternative *j*	35.425375
31	voter06	alternative *b*	5.490934
32	voter06	alternative *e*	5.490934
33	voter07	alternative *b*	22.855333
34	voter07	alternative *j*	22.855333
35	voter08	alternative *b*	19.835570
36	voter09	alternative *c*	22.928716
37	voter09	alternative *e*	22.928716
38	voter09	alternative *j*	22.928716
39	voter10	alternative *c*	5.538309
40	voter10	alternative *e*	5.538309
41	voter11	alternative *c*	13.130227
42	voter11	alternative *j*	13.130227
43	voter12	alternative *c*	6.056291
44	voter13	alternative *e*	23.992772
45	voter13	alternative *j*	23.992772
46	voter14	alternative *e*	16.699207
47	voter15	alternative *j*	98.165759
48	alternative *b*	drain	$r^{(z-1)}$
49	alternative *c*	drain	$r^{(z-1)}$
50	alternative *e*	drain	$r^{(z-1)}$
51	alternative *j*	drain	$r^{(z-1)}$

The following 13 pages document the solutions for (9.2.2.5) – (9.2.2.10).

We get:

$r^{(0)} = 82.583893$; $s^{(0)} = 0.000000$
$r^{(1)} = 78.688426$; $s^{(1)} = 71.469640$
$r^{(2)} = 77.714559$; $s^{(2)} = 75.365107$
$r^{(3)} = 77.471093$; $s^{(3)} = 76.740693$
$r^{(4)} = 77.410226$; $s^{(4)} = 77.227626$
$r^{(5)} = 77.395009$; $s^{(5)} = 77.349359$
$r^{(6)} = 77.391205$; $s^{(6)} = 77.379793$
$r^{(7)} = 77.390254$; $s^{(7)} = 77.387401$
$r^{(8)} = 77.390016$; $s^{(8)} = 77.389303$
$r^{(9)} = 77.389957$; $s^{(9)} = 77.389779$
$r^{(10)} = 77.389942$; $s^{(10)} = 77.389897$
$r^{(11)} = 77.389938$; $s^{(11)} = 77.389927$
$r^{(12)} = 77.389937$; $s^{(12)} = 77.389935$

We get:

$$r = \lim_{z \to \infty} r^{(z)} = \lim_{z \to \infty} s^{(z)} = 77.389937$$

Stage $z = 1$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	82.583893
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	34.239372
18	voter01	alternative *e*	36.597383	2.358011
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	35.425375
29	voter05	alternative *e*	35.425375	0.000000
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	82.583893
48	alternative *b*	drain	$r^{(0)} = 82.583893$	78.116279
49	alternative *c*	drain	$r^{(0)} = 82.583893$	82.583893
50	alternative *e*	drain	$r^{(0)} = 82.583893$	71.469640
51	alternative *j*	drain	$r^{(0)} = 82.583893$	82.583893

$r^{(1)} = (78.116279 + 82.583893 + 71.469640 + 82.583893) / 4 = 78.688426$
$s^{(1)} = \max \{ 0.000000; \min \{ 78.116279; 82.583893; 71.469640; 82.583893 \} \} = 71.469640$

Stage $z = 2$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	78.688426
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	30.343905
18	voter01	alternative *e*	36.597383	6.253478
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	35.425375
29	voter05	alternative *e*	35.425375	0.000000
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	78.688426
48	alternative *b*	drain	$r^{(1)} = 78.688426$	78.116279
49	alternative *c*	drain	$r^{(1)} = 78.688426$	78.688426
50	alternative *e*	drain	$r^{(1)} = 78.688426$	75.365107
51	alternative *j*	drain	$r^{(1)} = 78.688426$	78.688426

$r^{(2)} = (78.116279 + 78.688426 + 75.365107 + 78.688426) / 4 = 77.714559$
$s^{(2)} = \max \{ 71.469640; \min \{ 78.116279; 78.688426; 75.365107 ; 78.688426 \} \} = 75.365107$

Stage $z = 3$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.714559
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.370038
18	voter01	alternative *e*	36.597383	7.227344
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	35.023656
29	voter05	alternative *e*	35.425375	0.401719
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.714559
48	alternative *b*	drain	$r^{(2)} = 77.714559$	77.714559
49	alternative *c*	drain	$r^{(2)} = 77.714559$	77.714559
50	alternative *e*	drain	$r^{(2)} = 77.714559$	76.740693
51	alternative *j*	drain	$r^{(2)} = 77.714559$	77.714559

$r^{(3)} = (77.714559 + 77.714559 + 76.740693 + 77.714559) / 4 = 77.471093$
$s^{(3)} = \max \{ 75.365107; \min \{ 77.714559; 77.714559; 76.740693 ; 77.714559 \} \} = 76.740693$

Stage z = 4:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.471093
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.126572
18	voter01	alternative *e*	36.597383	7.470811
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.780190
29	voter05	alternative *e*	35.425375	0.645186
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.471093
48	alternative *b*	drain	$r^{(3)}$ = 77.471093	77.471093
49	alternative *c*	drain	$r^{(3)}$ = 77.471093	77.471093
50	alternative *e*	drain	$r^{(3)}$ = 77.471093	77.227626
51	alternative *j*	drain	$r^{(3)}$ = 77.471093	77.471093

$r^{(4)}$ = (77.471093 + 77.471093 + 77.227626 + 77.471093) / 4 = 77.410226
$s^{(4)}$ = max { 76.740693; min { 77.471093; 77.471093; 77.227626 ; 77.471093 } } = 77.227626

Stage z = 5:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.410226
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.065705
18	voter01	alternative *e*	36.597383	7.531678
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.719323
29	voter05	alternative *e*	35.425375	0.706053
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.410226
48	alternative *b*	drain	$r^{(4)}$ = 77.410226	77.410226
49	alternative *c*	drain	$r^{(4)}$ = 77.410226	77.410226
50	alternative *e*	drain	$r^{(4)}$ = 77.410226	77.349359
51	alternative *j*	drain	$r^{(4)}$ = 77.410226	77.410226

$r^{(5)}$ = (77.410226 + 77.410226 + 77.349359 + 77.410226) / 4 = 77.395009
$s^{(5)}$ = max { 77.227626; min { 77.410226; 77.410226; 77.349359; 77.410226 } } = 77.349359

Stage z = 6:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.395009
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.050488
18	voter01	alternative *e*	36.597383	7.546894
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.704106
29	voter05	alternative *e*	35.425375	0.721269
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.395009
48	alternative *b*	drain	$r^{(5)}$ = 77.395009	77.395009
49	alternative *c*	drain	$r^{(5)}$ = 77.395009	77.395009
50	alternative *e*	drain	$r^{(5)}$ = 77.395009	77.379793
51	alternative *j*	drain	$r^{(5)}$ = 77.395009	77.395009

$r^{(6)}$ = (77.395009 + 77.395009 + 77.379793 + 77.395009) / 4 = 77.391205
$s^{(6)}$ = max { 77.349359; min { 77.395009; 77.395009; 77.379793; 77.395009 } } = 77.379793

Stage $z = 7$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.391205
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.046684
18	voter01	alternative *e*	36.597383	7.550699
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.700302
29	voter05	alternative *e*	35.425375	0.725073
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.391205
48	alternative *b*	drain	$r^{(6)} = 77.391205$	77.391205
49	alternative *c*	drain	$r^{(6)} = 77.391205$	77.391205
50	alternative *e*	drain	$r^{(6)} = 77.391205$	77.387401
51	alternative *j*	drain	$r^{(6)} = 77.391205$	77.391205

$r^{(7)} = (77.391205 + 77.391205 + 77.387401 + 77.391205) / 4 = 77.390254$
$s^{(7)} = \max \{ 77.379793; \min \{ 77.391205; 77.391205; 77.387401; 77.391205 \} \} = 77.387401$

Stage z = 8:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.390254
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045733
18	voter01	alternative *e*	36.597383	7.551650
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699351
29	voter05	alternative *e*	35.425375	0.726024
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.390254
48	alternative *b*	drain	$r^{(7)}$ = 77.390254	77.390254
49	alternative *c*	drain	$r^{(7)}$ = 77.390254	77.390254
50	alternative *e*	drain	$r^{(7)}$ = 77.390254	77.389303
51	alternative *j*	drain	$r^{(7)}$ = 77.390254	77.390254

$r^{(8)}$ = (77.390254 + 77.390254 + 77.389303 + 77.390254) / 4 = 77.390016
$s^{(8)}$ = max { 77.387401; min { 77.390254; 77.390254; 77.389303; 77.390254 } } = 77.389303

Stage z = 9:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.390016
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045495
18	voter01	alternative *e*	36.597383	7.551887
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699113
29	voter05	alternative *e*	35.425375	0.726262
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.390016
48	alternative *b*	drain	$r^{(8)}$ = 77.390016	77.390016
49	alternative *c*	drain	$r^{(8)}$ = 77.390016	77.390016
50	alternative *e*	drain	$r^{(8)}$ = 77.390016	77.389779
51	alternative *j*	drain	$r^{(8)}$ = 77.390016	77.390016

$r^{(9)}$ = (77.390016 + 77.390016 + 77.389779 + 77.390016) / 4 = 77.389957
$s^{(9)}$ = max { 77.389303; min { 77.390016; 77.390016; 77.389779; 77.390016 } } = 77.389779

Stage z = 10:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389957
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045436
18	voter01	alternative *e*	36.597383	7.551947
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699054
29	voter05	alternative *e*	35.425375	0.726322
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.389957
48	alternative *b*	drain	$r^{(9)}$ = 77.389957	77.389957
49	alternative *c*	drain	$r^{(9)}$ = 77.389957	77.389957
50	alternative *e*	drain	$r^{(9)}$ = 77.389957	77.389897
51	alternative *j*	drain	$r^{(9)}$ = 77.389957	77.389957

$r^{(10)}$ = (77.389957 + 77.389957 + 77.389897 + 77.389957) / 4 = 77.389942
$s^{(10)}$ = max { 77.389779; min { 77.389957; 77.389957; 77.389897; 77.389957 } } = 77.389897

Stage z = 11:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389942
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045421
18	voter01	alternative *e*	36.597383	7.551962
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699039
29	voter05	alternative *e*	35.425375	0.726336
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.389942
48	alternative *b*	drain	$r^{(10)}$ = 77.389942	77.389942
49	alternative *c*	drain	$r^{(10)}$ = 77.389942	77.389942
50	alternative *e*	drain	$r^{(10)}$ = 77.389942	77.389927
51	alternative *j*	drain	$r^{(10)}$ = 77.389942	77.389942

$r^{(11)}$ = (77.389942 + 77.389942 + 77.389927 + 77.389942) / 4 = 77.389938
$s^{(11)}$ = max { 77.389897; min { 77.389942; 77.389942; 77.389927; 77.389942 } } = 77.389927

Stage $z = 12$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389938
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045417
18	voter01	alternative *e*	36.597383	7.551965
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699035
29	voter05	alternative *e*	35.425375	0.726340
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.389938
48	alternative *b*	drain	$r^{(11)} = 77.389938$	77.389938
49	alternative *c*	drain	$r^{(11)} = 77.389938$	77.389938
50	alternative *e*	drain	$r^{(11)} = 77.389938$	77.389935
51	alternative *j*	drain	$r^{(11)} = 77.389938$	77.389938

$r^{(12)} = (77.389938 + 77.389938 + 77.389935 + 77.389938) / 4 = 77.389937$
$s^{(12)} = \max \{ 77.389927; \min \{ 77.389938; 77.389938; 77.389935; 77.389938 \} \} = 77.389935$

Stage z = 13:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389937
16	voter01	alternative *b*	36.597383	0.000000
17	voter01	alternative *c*	36.597383	29.045416
18	voter01	alternative *e*	36.597383	7.551966
19	voter01	alternative *j*	36.597383	0.000000
20	voter02	alternative *b*	5.481150	0.000000
21	voter02	alternative *c*	5.481150	5.481150
22	voter02	alternative *e*	5.481150	0.000000
23	voter03	alternative *b*	13.279131	0.000000
24	voter03	alternative *c*	13.279131	13.279131
25	voter03	alternative *j*	13.279131	0.000000
26	voter04	alternative *b*	4.859413	0.000000
27	voter04	alternative *c*	4.859413	4.859413
28	voter05	alternative *b*	35.425375	34.699034
29	voter05	alternative *e*	35.425375	0.726341
30	voter05	alternative *j*	35.425375	0.000000
31	voter06	alternative *b*	5.490934	0.000000
32	voter06	alternative *e*	5.490934	5.490934
33	voter07	alternative *b*	22.855333	22.855333
34	voter07	alternative *j*	22.855333	0.000000
35	voter08	alternative *b*	19.835570	19.835570
36	voter09	alternative *c*	22.928716	0.000000
37	voter09	alternative *e*	22.928716	22.928716
38	voter09	alternative *j*	22.928716	0.000000
39	voter10	alternative *c*	5.538309	5.538309
40	voter10	alternative *e*	5.538309	0.000000
41	voter11	alternative *c*	13.130227	13.130227
42	voter11	alternative *j*	13.130227	0.000000
43	voter12	alternative *c*	6.056291	6.056291
44	voter13	alternative *e*	23.992772	23.992772
45	voter13	alternative *j*	23.992772	0.000000
46	voter14	alternative *e*	16.699207	16.699207
47	voter15	alternative *j*	98.165759	77.389937
48	alternative *b*	drain	$r^{(12)}$ = 77.389937	77.389937
49	alternative *c*	drain	$r^{(12)}$ = 77.389937	77.389937
50	alternative *e*	drain	$r^{(12)}$ = 77.389937	77.389936
51	alternative *j*	drain	$r^{(12)}$ = 77.389937	77.389937

$r^{(13)}$ = (77.389937 + 77.389937 + 77.389936 + 77.389937) / 4 = 77.389937
$s^{(13)}$ = max { 77.389935; min { 77.389937; 77.389937; 77.389936; 77.389937 } } = 77.389936

9.2.3. Applying the Schulze Tie-Breaking Method

Table 9.2.3.1 lists the links in example A53.

11	*a*	*b*	*c*	*e*	*j*	77.389937	99.563758	**107.281879**	101.107383	69.463087

For example, row 11 of table 9.2.3.1 contains the following information:

- $N[\{b,c,e,j\};a] = 77.389937$
- $N[\{a,c,e,j\};b] = 99.563758$
- $N[\{a,b,e,j\};c] = 107.281879$
- $N[\{a,b,c,j\};e] = 101.107383$
- $N[\{a,b,c,e\};j] = 69.463087$

- The link $\{b, c, e, j\} \rightarrow \{a, c, e, j\}$ has a strength of $(N[\{b,c,e,j\};a], N[\{a,c,e,j\};b])$.
- The link $\{b, c, e, j\} \rightarrow \{a, b, e, j\}$ has a strength of $(N[\{b,c,e,j\};a], N[\{a,b,e,j\};c])$.
- The link $\{b, c, e, j\} \rightarrow \{a, b, c, j\}$ has a strength of $(N[\{b,c,e,j\};a], N[\{a,b,c,j\};e])$.
- The link $\{b, c, e, j\} \rightarrow \{a, b, c, e\}$ has a strength of $(N[\{b,c,e,j\};a], N[\{a,b,c,e\};j])$.

- The link $\{a, c, e, j\} \rightarrow \{b, c, e, j\}$ has a strength of $(N[\{a,c,e,j\};b], N[\{b,c,e,j\};a])$.
- The link $\{a, c, e, j\} \rightarrow \{a, b, e, j\}$ has a strength of $(N[\{a,c,e,j\};b], N[\{a,b,e,j\};c])$.
- The link $\{a, c, e, j\} \rightarrow \{a, b, c, j\}$ has a strength of $(N[\{a,c,e,j\};b], N[\{a,b,c,j\};e])$.
- The link $\{a, c, e, j\} \rightarrow \{a, b, c, e\}$ has a strength of $(N[\{a,c,e,j\};b], N[\{a,b,c,e\};j])$.

- The link $\{a, b, e, j\} \rightarrow \{b, c, e, j\}$ has a strength of $(N[\{a,b,e,j\};c], N[\{b,c,e,j\};a])$.
- The link $\{a, b, e, j\} \rightarrow \{a, c, e, j\}$ has a strength of $(N[\{a,b,e,j\};c], N[\{a,c,e,j\};b])$.
- The link $\{a, b, e, j\} \rightarrow \{a, b, c, j\}$ has a strength of $(N[\{a,b,e,j\};c], N[\{a,b,c,j\};e])$.
- The link $\{a, b, e, j\} \rightarrow \{a, b, c, e\}$ has a strength of $(N[\{a,b,e,j\};c], N[\{a,b,c,e\};j])$.

- The link $\{a, b, c, j\} \rightarrow \{b, c, e, j\}$ has a strength of $(N[\{a,b,c,j\};e], N[\{b,c,e,j\};a])$.
- The link $\{a, b, c, j\} \rightarrow \{a, c, e, j\}$ has a strength of $(N[\{a,b,c,j\};e], N[\{a,c,e,j\};b])$.
- The link $\{a, b, c, j\} \rightarrow \{a, b, e, j\}$ has a strength of $(N[\{a,b,c,j\};e], N[\{a,b,e,j\};c])$.
- The link $\{a, b, c, j\} \rightarrow \{a, b, c, e\}$ has a strength of $(N[\{a,b,c,j\};e], N[\{a,b,c,e\};j])$.

- The link $\{a, b, c, e\} \rightarrow \{b, c, e, j\}$ has a strength of $(N[\{a,b,c,e\};j], N[\{b,c,e,j\};a])$.
- The link $\{a, b, c, e\} \rightarrow \{a, c, e, j\}$ has a strength of $(N[\{a,b,c,e\};j], N[\{a,c,e,j\};b])$.
- The link $\{a, b, c, e\} \rightarrow \{a, b, e, j\}$ has a strength of $(N[\{a,b,c,e\};j], N[\{a,b,e,j\};c])$.
- The link $\{a, b, c, e\} \rightarrow \{a, b, c, j\}$ has a strength of $(N[\{a,b,c,e\};j], N[\{a,b,c,j\};e])$.

When we apply the Schulze tie-breaker, as defined at stage 3 of section 9.1.3, to the links of table 9.2.3.1, we get $\{a, d, g, j\}$ as winning set.

For example, we have:

- Line 33: The link $\{a, b, g, j\} \rightarrow \{a, d, g, j\}$ has a strength of $(N[\{a,b,g,j\};d], N[\{a,d,g,j\};b])$ = (101.411379, 102.166302).
- Line 33: The link $\{a, d, g, j\} \rightarrow \{a, b, g, j\}$ has a strength of $(N[\{a,d,g,j\};b], N[\{a,b,g,j\};d])$ = (102.166302, 101.411379).
- Line 49: The link $\{a, b, g, j\} \rightarrow \{a, f, g, j\}$ has a strength of $(N[\{a,b,g,j\};f], N[\{a,f,g,j\};b])$ = (101.068282, 102.334802).
- Line 49: The link $\{a, f, g, j\} \rightarrow \{a, b, g, j\}$ has a strength of $(N[\{a,f,g,j\};b], N[\{a,b,g,j\};f])$ = (102.334802, 101.068282).
- Line 104: The link $\{a, d, g, j\} \rightarrow \{a, f, g, j\}$ has a strength of $(N[\{a,d,g,j\};f], N[\{a,f,g,j\};d])$ = (101.351648, 101.098901).
- Line 104: The link $\{a, f, g, j\} \rightarrow \{a, d, g, j\}$ has a strength of $(N[\{a,f,g,j\};d], N[\{a,d,g,j\};f])$ = (101.098901, 101.351648).

So $\{a, d, g, j\}$ beats $\{a, b, g, j\}$ in the direct comparison, $\{a, f, g, j\}$ beats $\{a, b, g, j\}$ in the direct comparison, and $\{a, d, g, j\}$ beats $\{a, f, g, j\}$ in the direct comparison.

When there are C alternatives, then there are $(C!)/(((M+1)!)\cdot((C-M-1)!))$ possible $(M+1)$-way contests. For $C = 10$ and $M = 4$, we get 252 possible 5-way contests. Table 9.2.3.1 lists these 252 possible 5-way contests for example A53.

When Schulze STV is used to choose M from $(M+1)$ alternatives $\{a_1,...,a_{(M+1)}\}$, then that alternative $k \in \{1,...,(M+1)\}$ is eliminated for which $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$ is the maximum, while the other M alternatives are elected. In table 9.2.3.1, the maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$ of each 5-way contest is **fat and underlined**.

Suppose the maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$ of a $(M+1)$-way contest is not unique. Suppose $1 < m \leq (M+1)$ entries are tied for maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$, then the m alternatives with maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$ are tied for winning one of the remaining $(m-1)$ seats, while the other $(M+1-m)$ alternatives are elected. In table 9.2.3.1 for those 5-way contests, where the maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$ is not unique, those $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$, that are tied for maximum $N[(\{a_1,...,a_{(M+1)}\}\backslash\{a_k\});a_k]$, are *italic and underlined* (only lines 27, 149, and 155).

In table 9.2.3.1, we see:

- Alternatives *a*, *g*, and *j* each win in every 5-way contest.
- Alternative *d* is tied for winning in one 5-way contest (line 27) and wins in every other 5-way contest.
- Alternative *f* loses in one 5-way contest (line 104) and wins in every other 5-way contest.
- Alternative *b* wins in 121 5-way contests, is tied for winning in one 5-way contest (line 27), and loses in four 5-way contests (lines 30, 33, 49, and 174).
- Alternative *e* wins 111 times and loses 15 times.
- Alternative *h* wins 59 times and loses 67 times.
- Alternative *c* wins 45 times, is tied twice (lines 149 and 155), and loses 79 times.
- Alternative *i* wins 41 times, is tied twice (lines 149 and 155), and loses 83 times.

	k	*l*	*m*	*n*	*o*	*N*[{*l*,*m*,*n*,*o*};*k*]	*N*[{*k*,*m*,*n*,*o*};*l*]	*N*[{*k*,*l*,*n*,*o*};*m*]	*N*[{*k*,*l*,*m*,*o*};*n*]	*N*[{*k*,*l*,*m*,*n*};*o*]
1	*a*	*b*	*c*	*d*	*e*	69.311512	97.347630	**104.356659**	91.117381	97.866817
2	*a*	*b*	*c*	*d*	*f*	72.494331	97.267574	**106.394558**	91.791383	92.052154
3	*a*	*b*	*c*	*d*	*g*	74.292035	97.699115	**105.077434**	97.444690	85.486726
4	*a*	*b*	*c*	*d*	*h*	69.482146	95.615034	**103.473804**	90.375854	101.053161
5	*a*	*b*	*c*	*d*	*i*	68.329596	95.403587	**103.912556**	91.535874	100.818386
6	*a*	*b*	*c*	*d*	*j*	83.765432	100.720621	**106.330377**	96.895787	70.631929
7	*a*	*b*	*c*	*e*	*f*	68.050459	96.800459	**106.559633**	97.327982	91.261468
8	*a*	*b*	*c*	*e*	*g*	71.971047	98.864143	**106.035635**	98.608018	84.521158
9	*a*	*b*	*c*	*e*	*h*	65.248069	95.126728	**104.665899**	95.391705	99.567599
10	*a*	*b*	*c*	*e*	*i*	63.064516	95.126728	**104.400922**	96.186636	101.221198
11	*a*	*b*	*c*	*e*	*j*	77.389937	99.563758	**107.281879**	101.107383	69.463087
12	*a*	*b*	*c*	*f*	*g*	73.393258	98.202247	**107.505618**	95.101124	85.797753
13	*a*	*b*	*c*	*f*	*h*	68.320236	95.877598	**105.704388**	88.972286	101.125492
14	*a*	*b*	*c*	*f*	*i*	65.979263	94.596774	**106.255760**	92.741935	100.426267
15	*a*	*b*	*c*	*f*	*j*	82.285264	100.495495	**107.229730**	97.646396	72.004505
16	*a*	*b*	*c*	*g*	*h*	72.748673	96.828442	**106.173815**	81.252822	102.996248
17	*a*	*b*	*c*	*g*	*i*	70.450450	96.869369	**105.675676**	83.141892	103.862613
18	*a*	*b*	*c*	*g*	*j*	86.629956	102.334802	**108.667401**	88.403084	73.964758
19	*a*	*b*	*c*	*h*	*i*	63.805224	93.221709	**103.845266**	99.797547	99.330254
20	*a*	*b*	*c*	*h*	*j*	76.937668	98.977528	105.438202	**108.022472**	67.449438
21	*a*	*b*	*c*	*i*	*j*	75.764706	99.529148	**106.233184**	105.201794	67.719298
22	*a*	*b*	*d*	*e*	*f*	74.020045	97.839644	92.973274	**100.913140**	94.253898
23	*a*	*b*	*d*	*e*	*g*	75.571429	99.329670	97.813187	**100.846154**	86.439560
24	*a*	*b*	*d*	*e*	*h*	70.771762	97.646396	91.430180	98.423423	**101.728238**
25	*a*	*b*	*d*	*e*	*i*	69.205817	96.733781	92.360179	99.049217	**102.651007**
26	*a*	*b*	*d*	*e*	*j*	86.821192	100.529801	97.483444	**102.814570**	72.350993
27	*a*	*b*	*d*	*f*	*g*	77.090708	*98.716814*	*98.716814*	97.444690	88.030973
28	*a*	*b*	*d*	*f*	*h*	74.397888	98.164414	91.948198	92.725225	**102.764274**
29	*a*	*b*	*d*	*f*	*i*	72.322222	96.600000	93.277778	95.833333	**101.966667**
30	*a*	*b*	*d*	*f*	*j*	87.716186	**100.975610**	96.895787	99.190687	75.221729
31	*a*	*b*	*d*	*g*	*h*	76.388633	98.462389	96.681416	83.960177	**104.507385**
32	*a*	*b*	*d*	*g*	*i*	73.946785	97.660754	97.915743	85.421286	**105.055432**
33	*a*	*b*	*d*	*g*	*j*	89.332604	**102.166302**	101.411379	90.842451	76.247265
34	*a*	*b*	*d*	*h*	*i*	69.217708	96.092342	91.430180	101.469229	**101.790541**
35	*a*	*b*	*d*	*h*	*j*	84.333333	100.433333	95.577778	**108.100000**	71.555556
36	*a*	*b*	*d*	*i*	*j*	84.176158	100.243363	96.935841	**106.095133**	72.256637
37	*a*	*b*	*e*	*f*	*g*	75.055310	99.734513	**100.243363**	97.444690	87.522124
38	*a*	*b*	*e*	*f*	*h*	70.311453	97.307692	98.088235	92.104072	**102.188547**
39	*a*	*b*	*e*	*f*	*i*	67.847380	95.876993	97.972665	95.091116	**103.211845**
40	*a*	*b*	*e*	*f*	*j*	84.966518	99.598214	**101.651786**	98.828125	74.955357
41	*a*	*b*	*e*	*g*	*h*	74.337778	99.120267	99.632517	82.984410	**103.925029**
42	*a*	*b*	*e*	*g*	*i*	72.131696	98.828125	99.598214	84.196429	**105.245536**
43	*a*	*b*	*e*	*g*	*j*	88.208791	101.351648	**104.131868**	90.230769	76.076923
44	*a*	*b*	*e*	*h*	*i*	64.914754	95.308219	96.358447	101.021319	**102.397260**
45	*a*	*b*	*e*	*h*	*j*	81.744689	98.828125	100.111607	**107.555804**	71.104911
46	*a*	*b*	*e*	*i*	*j*	78.449612	99.306488	100.850112	**107.281879**	70.492170
47	*a*	*b*	*f*	*g*	*h*	76.384893	99.529148	94.630045	84.831839	**104.624076**
48	*a*	*b*	*f*	*g*	*i*	73.671875	98.058036	97.287946	85.993304	**104.988839**
49	*a*	*b*	*f*	*g*	*j*	87.643172	**102.334802**	101.068282	90.429515	78.524229
50	*a*	*b*	*f*	*h*	*i*	68.484353	95.000000	92.105263	**102.305121**	102.105263

Table 9.2.3.1 (part 1 of 5): links in example A53

	k	*l*	*m*	*n*	*o*	$N[\{l,m,n,o\};k]$	$N[\{k,m,n,o\};l]$	$N[\{k,l,n,o\};m]$	$N[\{k,l,m,o\};n]$	$N[\{k,l,m,n\};o]$
51	*a*	*b*	*f*	*h*	*j*	82.438202	99.752809	95.876404	**109.056180**	72.876404
52	*a*	*b*	*f*	*i*	*j*	82.769058	99.529148	97.724215	**106.748879**	73.228700
53	*a*	*b*	*g*	*h*	*i*	73.008267	97.347630	81.512415	103.255842	**104.875847**
54	*a*	*b*	*g*	*h*	*j*	86.441242	101.230599	86.951220	**110.410200**	74.966741
55	*a*	*b*	*g*	*i*	*j*	86.377778	101.966667	88.677778	**109.122222**	73.855556
56	*a*	*b*	*h*	*i*	*j*	78.376979	98.608597	**107.454751**	104.852941	68.028169
57	*a*	*c*	*d*	*e*	*f*	72.781532	**106.452703**	91.430180	97.128378	92.207207
58	*a*	*c*	*d*	*e*	*g*	74.635762	**104.845475**	96.975717	97.737307	85.805740
59	*a*	*c*	*d*	*e*	*h*	69.115667	**104.235160**	90.582192	93.995434	102.071548
60	*a*	*c*	*d*	*e*	*i*	67.753950	**103.837472**	91.117381	95.011287	102.279910
61	*a*	*c*	*d*	*e*	*j*	84.830247	**107.095344**	97.405765	100.465632	69.866962
62	*a*	*c*	*d*	*f*	*g*	74.698661	**107.299107**	97.544643	94.720982	85.736607
63	*a*	*c*	*d*	*f*	*h*	71.415141	**105.329545**	90.693182	91.215909	101.346222
64	*a*	*c*	*d*	*f*	*i*	69.728507	**105.893665**	90.542986	93.404977	100.429864
65	*a*	*c*	*d*	*f*	*j*	86.314607	**106.988764**	96.134831	98.460674	72.101124
66	*a*	*c*	*d*	*g*	*h*	73.476924	**104.988839**	95.747768	81.629464	104.157004
67	*a*	*c*	*d*	*g*	*i*	71.361607	**104.475446**	96.517857	83.939732	103.705357
68	*a*	*c*	*d*	*g*	*j*	87.389868	**108.414097**	101.574890	88.909692	73.711454
69	*a*	*c*	*d*	*h*	*i*	66.115932	**102.894737**	90.000000	100.463016	100.526316
70	*a*	*c*	*d*	*h*	*j*	81.284987	105.245536	96.004464	**107.555804**	67.767857
71	*a*	*c*	*d*	*i*	*j*	80.402166	**105.995526**	96.219239	105.480984	69.205817
72	*a*	*c*	*e*	*f*	*g*	72.833333	**107.588889**	97.877778	95.066667	86.633333
73	*a*	*c*	*e*	*f*	*h*	67.058858	**106.295872**	94.690367	90.206422	101.748481
74	*a*	*c*	*e*	*f*	*i*	64.303944	**105.928074**	94.454756	92.053364	103.259861
75	*a*	*c*	*e*	*f*	*j*	81.705790	**107.247191**	99.752809	97.943820	73.134831
76	*a*	*c*	*e*	*g*	*h*	71.904859	**106.471910**	96.134831	81.921348	103.567051
77	*a*	*c*	*e*	*g*	*i*	69.775281	**105.438202**	96.393258	83.730337	104.662921
78	*a*	*c*	*e*	*g*	*j*	85.428571	**108.934066**	102.615385	88.967033	74.054945
79	*a*	*c*	*e*	*h*	*i*	61.699912	**104.060325**	92.053364	100.794287	101.392111
80	*a*	*c*	*e*	*h*	*j*	76.021251	106.510067	98.020134	**107.796421**	66.018519
81	*a*	*c*	*e*	*i*	*j*	72.631579	106.693002	98.905192	**106.952596**	65.612403
82	*a*	*c*	*f*	*g*	*h*	72.163286	**107.764045**	93.292135	82.696629	104.083905
83	*a*	*c*	*f*	*g*	*i*	69.414414	**107.747748**	94.538288	84.177928	104.121622
84	*a*	*c*	*f*	*g*	*j*	84.911308	**109.135255**	100.465632	88.481153	77.006652
85	*a*	*c*	*f*	*h*	*i*	62.378284	**105.372093**	90.662791	100.761251	100.825581
86	*a*	*c*	*f*	*h*	*j*	77.297595	106.433409	96.049661	**107.731377**	70.349887
87	*a*	*c*	*f*	*i*	*j*	75.027712	**106.394558**	97.267574	106.133787	70.147392
88	*a*	*c*	*g*	*h*	*i*	68.278778	**105.852273**	79.454545	102.914404	103.500000
89	*a*	*c*	*g*	*h*	*j*	84.077778	108.611111	84.844444	**109.888889**	72.577778
90	*a*	*c*	*g*	*i*	*j*	82.672811	**108.866667**	86.888889	108.355556	72.066667
91	*a*	*c*	*h*	*i*	*j*	69.181244	105.351474	**106.655329**	104.308390	63.411215
92	*a*	*d*	*e*	*f*	*g*	77.087912	98.318681	**99.329670**	97.054945	88.208791
93	*a*	*d*	*e*	*f*	*h*	73.290722	92.567265	97.724215	92.567265	**103.850533**
94	*a*	*d*	*e*	*f*	*i*	71.521253	92.617450	96.733781	94.932886	**104.194631**
95	*a*	*d*	*e*	*f*	*j*	87.144444	97.366667	**101.200000**	99.155556	75.133333
96	*a*	*d*	*e*	*g*	*h*	75.618401	96.559020	98.351893	84.008909	**105.461777**
97	*a*	*d*	*e*	*g*	*i*	73.691796	97.915743	97.405765	85.166297	**105.820399**
98	*a*	*d*	*e*	*g*	*j*	88.829322	101.663020	**102.921225**	91.345733	75.240700
99	*a*	*d*	*e*	*h*	*i*	67.494687	91.477273	95.136364	102.653041	**103.238636**
100	*a*	*d*	*e*	*h*	*j*	84.656319	96.640798	99.700665	**108.370288**	70.631929

Table 9.2.3.1 (part 2 of 5): links in example A53

	k	*l*	*m*	*n*	*o*	$N[\{l,m,n,o\};k]$	$N[\{k,m,n,o\};l]$	$N[\{k,l,n,o\};m]$	$N[\{k,l,m,o\};n]$	$N[\{k,l,m,n\};o]$
101	*a*	*d*	*e*	*i*	*j*	83.348624	97.366667	100.688889	**107.333333**	70.533333
102	*a*	*d*	*f*	*g*	*h*	76.983694	97.111111	95.577778	85.100000	**105.227417**
103	*a*	*d*	*f*	*g*	*i*	74.366667	97.622222	96.600000	86.377778	**105.033333**
104	*a*	*d*	*f*	*g*	*j*	88.714286	101.098901	**101.351648**	90.736264	78.098901
105	*a*	*d*	*f*	*h*	*i*	70.051272	91.323529	93.665158	102.188547	**102.771493**
106	*a*	*d*	*f*	*h*	*j*	84.966518	95.491071	97.544643	**108.325893**	73.671875
107	*a*	*d*	*f*	*i*	*j*	85.223214	96.261161	98.571429	**106.785714**	73.158482
108	*a*	*d*	*g*	*h*	*i*	73.032875	96.177130	81.737668	104.108381	**104.943946**
109	*a*	*d*	*g*	*h*	*j*	87.197802	100.087912	87.956044	**110.197802**	74.560440
110	*a*	*d*	*g*	*i*	*j*	86.821192	101.291391	89.359823	**108.907285**	73.620309
111	*a*	*d*	*h*	*i*	*j*	80.164441	95.661435	**106.748879**	105.717489	69.876682
112	*a*	*e*	*f*	*g*	*h*	74.683694	98.900000	95.322222	85.611111	**105.482973**
113	*a*	*e*	*f*	*g*	*i*	72.131696	98.058036	97.031250	86.506696	**106.272321**
114	*a*	*e*	*f*	*g*	*j*	86.123348	**102.841410**	101.321586	91.189427	78.524229
115	*a*	*e*	*f*	*h*	*i*	65.476289	94.690367	92.052752	103.331050	**104.449541**
116	*a*	*e*	*f*	*h*	*j*	81.372768	98.828125	96.774554	**109.095982**	73.928571
117	*a*	*e*	*f*	*i*	*j*	80.965732	99.752809	97.943820	**108.539326**	72.617978
118	*a*	*e*	*g*	*h*	*i*	71.191111	97.088036	82.031603	104.553810	**105.135440**
119	*a*	*e*	*g*	*h*	*j*	85.486726	101.769912	88.030973	**110.674779**	74.037611
120	*a*	*e*	*g*	*i*	*j*	84.723451	102.533186	89.811947	**109.402655**	73.528761
121	*a*	*e*	*h*	*i*	*j*	74.210623	97.646396	**107.747748**	107.229730	66.179245
122	*a*	*f*	*g*	*h*	*i*	71.646432	94.842697	83.471910	104.600759	**105.438202**
123	*a*	*f*	*g*	*h*	*j*	84.656319	99.190687	87.461197	**110.665188**	78.026608
124	*a*	*f*	*g*	*i*	*j*	84.911308	99.955654	89.246120	**109.390244**	76.496674
125	*a*	*f*	*h*	*i*	*j*	77.294626	95.486425	**107.975113**	107.194570	70.509050
126	*a*	*g*	*h*	*i*	*j*	83.612975	85.671141	**109.597315**	108.568233	72.550336
127	*b*	*c*	*d*	*e*	*f*	89.066059	**101.116173**	86.708428	97.710706	85.398633
128	*b*	*c*	*d*	*e*	*g*	90.135135	**101.272523**	90.394144	97.387387	80.810811
129	*b*	*c*	*d*	*e*	*h*	88.255814	**97.616279**	82.906977	94.941860	96.279070
130	*b*	*c*	*d*	*e*	*i*	85.845070	97.453052	82.605634	95.833333	**98.262911**
131	*b*	*c*	*d*	*e*	*j*	97.877778	**103.755556**	92.255556	101.711111	64.400000
132	*b*	*c*	*d*	*f*	*g*	89.587054	**103.705357**	93.180804	90.100446	83.426339
133	*b*	*c*	*d*	*f*	*h*	90.057078	**99.771689**	85.593607	86.381279	98.196347
134	*b*	*c*	*d*	*f*	*i*	87.494305	**100.068337**	85.922551	87.861509	98.653298
135	*b*	*c*	*d*	*f*	*j*	98.864143	**104.755011**	93.229399	95.278396	67.873051
136	*b*	*c*	*d*	*g*	*h*	90.613839	**101.138393**	90.613839	78.292411	99.341518
137	*b*	*c*	*d*	*g*	*i*	88.542141	**101.116173**	90.113895	79.897494	100.330296
138	*b*	*c*	*d*	*g*	*j*	99.432314	**105.207424**	97.674672	88.133188	69.552402
139	*b*	*c*	*d*	*h*	*i*	86.516204	97.164352	82.256944	96.099537	**97.962963**
140	*b*	*c*	*d*	*h*	*j*	97.150776	101.995565	90.776053	**106.330377**	63.747228
141	*b*	*c*	*d*	*i*	*j*	97.190265	103.042035	91.084071	**105.586283**	63.097345
142	*b*	*c*	*e*	*f*	*g*	90.598194	**102.279910**	95.530474	88.521445	83.069977
143	*b*	*c*	*e*	*f*	*h*	88.211765	**98.764706**	92.000000	83.611765	97.411765
144	*b*	*c*	*e*	*f*	*i*	85.771971	98.337292	92.327791	84.242761	**99.320184**
145	*b*	*c*	*e*	*f*	*j*	96.828442	**104.356659**	99.943567	93.713318	65.158014
146	*b*	*c*	*e*	*g*	*h*	91.052632	**101.052632**	92.894737	77.105263	97.894737
147	*b*	*c*	*e*	*g*	*i*	88.790698	100.290698	92.802326	77.558140	**100.558140**
148	*b*	*c*	*e*	*g*	*j*	99.329670	**105.648352**	101.857143	86.692308	66.472527
149	*b*	*c*	*e*	*h*	*i*	84.166667	*96.111111*	88.888889	94.722222	*96.111111*
150	*b*	*c*	*e*	*h*	*j*	95.011287	102.799097	97.866817	**105.135440**	59.187359

Table 9.2.3.1 (part 3 of 5): links in example A53

	k	*l*	*m*	*n*	*o*	*N*[{*l*,*m*,*n*,*o*};*k*]	*N*[{*k*,*m*,*n*,*o*};*l*]	*N*[{*k*,*l*,*n*,*o*};*m*]	*N*[{*k*,*l*,*m*,*o*};*n*]	*N*[{*k*,*l*,*m*,*n*};*o*]
151	*b*	*c*	*e*	*i*	*j*	95.226244	103.031674	98.608597	**105.633484**	57.500000
152	*b*	*c*	*f*	*g*	*h*	91.578947	**101.842105**	87.631579	79.210526	99.736842
153	*b*	*c*	*f*	*g*	*i*	88.947368	**101.842105**	88.526779	80.263158	100.420590
154	*b*	*c*	*f*	*g*	*j*	99.835165	**105.901099**	96.549451	88.461538	69.252747
155	*b*	*c*	*f*	*h*	*i*	85.357995	*97.159905*	83.711217	96.610979	*97.159905*
156	*b*	*c*	*f*	*h*	*j*	96.049661	102.539503	91.117381	**106.693002**	63.600451
157	*b*	*c*	*f*	*i*	*j*	96.568849	103.058691	92.934537	**104.875847**	62.562077
158	*b*	*c*	*g*	*h*	*i*	87.909931	**100.392610**	74.896074	97.205543	99.595843
159	*b*	*c*	*g*	*h*	*j*	98.425721	104.290466	83.636364	**107.860310**	65.787140
160	*b*	*c*	*g*	*i*	*j*	99.006623	105.099338	85.044150	**106.876380**	63.973510
161	*b*	*c*	*h*	*i*	*j*	94.659864	100.918367	**105.351474**	102.743764	56.326531
162	*b*	*d*	*e*	*f*	*g*	91.338496	93.119469	**98.971239**	92.101770	84.469027
163	*b*	*d*	*e*	*f*	*h*	90.170455	85.727273	97.488636	87.818182	**98.795455**
164	*b*	*d*	*e*	*f*	*i*	87.879819	85.793651	96.746032	89.288441	**100.292058**
165	*b*	*d*	*e*	*f*	*j*	97.877778	93.533333	**102.477778**	96.088889	70.022222
166	*b*	*d*	*e*	*g*	*h*	92.258427	90.191011	97.943820	79.078652	**100.528090**
167	*b*	*d*	*e*	*g*	*i*	89.909091	90.431818	97.488636	79.715909	**102.454545**
168	*b*	*d*	*e*	*g*	*j*	99.146608	97.636761	**102.921225**	88.326039	71.969365
169	*b*	*d*	*e*	*h*	*i*	86.918605	82.104651	95.209302	96.279070	**99.488372**
170	*b*	*d*	*e*	*h*	*j*	97.111111	91.233333	100.688889	**105.288889**	65.677778
171	*b*	*d*	*e*	*i*	*j*	96.517857	91.127232	101.395089	**107.299107**	63.660714
172	*b*	*d*	*f*	*g*	*h*	93.180804	91.897321	91.897321	81.629464	**101.395089**
173	*b*	*d*	*f*	*g*	*i*	89.843750	91.897321	93.027237	82.399554	**102.832138**
174	*b*	*d*	*f*	*g*	*j*	**99.901532**	97.888403	99.146608	89.080963	73.982495
175	*b*	*d*	*f*	*h*	*i*	88.401361	84.229025	89.705215	97.528345	**100.136054**
176	*b*	*d*	*f*	*h*	*j*	98.133333	91.744444	94.300000	**106.311111**	69.511111
177	*b*	*d*	*f*	*i*	*j*	97.150776	92.560976	95.620843	**106.585366**	68.082040
178	*b*	*d*	*g*	*h*	*i*	90.449438	89.932584	78.044944	99.235955	**102.337079**
179	*b*	*d*	*g*	*h*	*j*	99.076923	96.043956	85.428571	**107.923077**	71.527473
180	*b*	*d*	*g*	*i*	*j*	98.788546	96.762115	86.629956	**108.414097**	69.405286
181	*b*	*d*	*h*	*i*	*j*	96.004464	89.843750	104.732143	**105.245536**	64.174107
182	*b*	*e*	*f*	*g*	*h*	92.784091	95.920455	90.170455	79.977273	**101.147727**
183	*b*	*e*	*f*	*g*	*i*	90.113895	95.353075	90.743058	80.683371	**103.106601**
184	*b*	*e*	*f*	*g*	*j*	99.295154	**102.081498**	98.535242	88.403084	71.685022
185	*b*	*e*	*f*	*h*	*i*	85.910165	90.531915	85.638298	97.872340	**100.047281**
186	*b*	*e*	*f*	*h*	*j*	96.049661	98.386005	92.934537	**105.914221**	66.715576
187	*b*	*e*	*f*	*i*	*j*	95.659091	99.318182	93.568182	**106.897727**	64.556818
188	*b*	*e*	*g*	*h*	*i*	90.357143	93.536866	75.518433	99.101382	**101.486175**
189	*b*	*e*	*g*	*h*	*j*	98.425721	100.465632	84.401330	**108.115299**	68.592018
190	*b*	*e*	*g*	*i*	*j*	98.462389	101.006637	85.486726	**108.639381**	66.404867
191	*b*	*e*	*h*	*i*	*j*	93.995434	96.358447	104.235160	**106.073059**	59.337900
192	*b*	*f*	*g*	*h*	*i*	90.526316	90.000000	77.631579	100.000000	**101.842105**
193	*b*	*f*	*g*	*h*	*j*	99.155556	95.322222	85.611111	**108.355556**	71.555556
194	*b*	*f*	*g*	*i*	*j*	99.155556	96.855556	86.377778	**107.844444**	69.766667
195	*b*	*f*	*h*	*i*	*j*	95.000000	90.789474	**105.789474**	105.263158	63.157895
196	*b*	*g*	*h*	*i*	*j*	97.982063	82.253363	**107.264574**	106.748879	65.751121
197	*c*	*d*	*e*	*f*	*g*	**102.193764**	91.180401	96.559020	88.106904	81.959911
198	*c*	*d*	*e*	*f*	*h*	**99.366359**	85.322581	94.066820	83.997696	97.246544
199	*c*	*d*	*e*	*f*	*i*	97.696759	84.386574	93.437500	84.227320	**100.251846**
200	*c*	*d*	*e*	*f*	*j*	**103.680089**	93.903803	100.335570	94.932886	67.147651

Table 9.2.3.1 (part 4 of 5): links in example A53

	k	l	m	n	o	$N[\{l,m,n,o\};k]$	$N[\{k,m,n,o\};l]$	$N[\{k,l,n,o\};m]$	$N[\{k,l,m,o\};n]$	$N[\{k,l,m,n\};o]$
201	*c*	*d*	*e*	*g*	*h*	**100.335570**	88.501119	95.190157	77.953020	98.020134
202	*c*	*d*	*e*	*g*	*i*	98.684211	87.894737	94.210526	78.157895	**101.052632**
203	*c*	*d*	*e*	*g*	*j*	**104.867841**	97.268722	102.081498	87.643172	68.138767
204	*c*	*d*	*e*	*h*	*i*	95.293427	81.525822	91.514085	93.943662	**97.723005**
205	*c*	*d*	*e*	*h*	*j*	102.165179	91.640625	98.571429	**105.758929**	61.863839
206	*c*	*d*	*e*	*i*	*j*	102.393736	91.588367	99.563758	**105.738255**	60.715884
207	*c*	*d*	*f*	*g*	*h*	**102.733333**	90.977778	88.677778	79.222222	98.388889
208	*c*	*d*	*f*	*g*	*i*	100.981941	89.559819	88.106552	79.954853	**101.396834**
209	*c*	*d*	*f*	*g*	*j*	**105.142857**	98.318681	97.560440	88.461538	70.516484
210	*c*	*d*	*f*	*h*	*i*	97.205543	83.660508	85.519630	94.815242	**98.799076**
211	*c*	*d*	*f*	*h*	*j*	102.881166	92.051570	93.598655	**105.201794**	66.266816
212	*c*	*d*	*f*	*i*	*j*	102.651007	92.617450	94.418345	**105.480984**	64.832215
213	*c*	*d*	*g*	*h*	*i*	99.200450	88.322072	75.889640	95.833333	**100.754505**
214	*c*	*d*	*g*	*h*	*j*	103.780488	96.385809	84.656319	**107.350333**	67.827051
215	*c*	*d*	*g*	*i*	*j*	104.314159	96.426991	86.250000	**106.603982**	66.404867
216	*c*	*d*	*h*	*i*	*j*	101.076233	90.246637	**104.428251**	103.654709	60.594170
217	*c*	*e*	*f*	*g*	*h*	**101.531532**	94.279279	86.509009	78.738739	98.941441
218	*c*	*e*	*f*	*g*	*i*	100.459770	92.793103	85.761385	78.517241	**102.468500**
219	*c*	*e*	*f*	*g*	*j*	**104.890110**	101.098901	97.054945	88.208791	68.747253
220	*c*	*e*	*f*	*h*	*i*	96.156627	88.674699	81.192771	95.602410	**98.373494**
221	*c*	*e*	*f*	*h*	*j*	102.482993	97.006803	91.791383	**106.133787**	62.585034
222	*c*	*e*	*f*	*i*	*j*	101.902050	98.234624	92.209567	**106.093394**	61.560364
223	*c*	*e*	*g*	*h*	*i*	99.438073	90.470183	74.380734	95.745413	**99.965596**
224	*c*	*e*	*g*	*h*	*j*	104.011111	99.411111	84.077778	**107.588889**	64.911111
225	*c*	*e*	*g*	*i*	*j*	104.059735	100.243363	85.486726	**106.858407**	63.351770
226	*c*	*e*	*h*	*i*	*j*	100.559361	95.570776	**104.497717**	103.710046	55.662100
227	*c*	*f*	*g*	*h*	*i*	**100.821918**	85.593607	76.141553	96.883562	100.559361
228	*c*	*f*	*g*	*h*	*j*	104.314159	94.391593	84.723451	**107.621681**	68.949115
229	*c*	*f*	*g*	*i*	*j*	104.314159	95.409292	86.504425	**106.858407**	66.913717
230	*c*	*f*	*h*	*i*	*j*	100.526316	90.526316	**105.000000**	103.947368	60.000000
231	*c*	*g*	*h*	*i*	*j*	102.935268	81.886161	**106.785714**	104.988839	63.404018
232	*d*	*e*	*f*	*g*	*h*	91.233333	97.366667	89.955556	80.755556	**100.688889**
233	*d*	*e*	*f*	*g*	*i*	91.073826	94.932886	90.148104	80.525727	**103.319458**
234	*d*	*e*	*f*	*g*	*j*	98.355263	**101.381579**	98.607456	89.024123	72.631579
235	*d*	*e*	*f*	*h*	*i*	83.926097	92.690531	86.050808	96.674365	**100.658199**
236	*d*	*e*	*f*	*h*	*j*	93.082960	98.755605	93.856502	**105.717489**	68.587444
237	*d*	*e*	*f*	*i*	*j*	93.082960	100.044843	94.114350	**107.006726**	65.751121
238	*d*	*e*	*g*	*h*	*i*	88.640449	95.101124	76.235955	98.202247	**101.820225**
239	*d*	*e*	*g*	*h*	*j*	95.918142	100.243363	85.741150	**108.639381**	69.457965
240	*d*	*e*	*g*	*i*	*j*	96.508811	100.814978	86.883260	**107.907489**	67.885463
241	*d*	*e*	*h*	*i*	*j*	90.394144	97.646396	104.380631	**105.416667**	62.162162
242	*d*	*f*	*g*	*h*	*i*	89.587054	90.100446	78.805804	98.828125	**102.678571**
243	*d*	*f*	*g*	*h*	*j*	95.960265	96.214128	86.567329	**107.891832**	73.366446
244	*d*	*f*	*g*	*i*	*j*	97.015419	96.762115	87.389868	**107.907489**	70.925110
245	*d*	*f*	*h*	*i*	*j*	91.171171	92.725225	103.862613	**105.934685**	66.306306
246	*d*	*g*	*h*	*i*	*j*	94.600887	83.891353	**106.585366**	106.330377	68.592018
247	*e*	*f*	*g*	*h*	*i*	93.568182	88.079545	76.840909	99.318182	**102.193182**
248	*e*	*f*	*g*	*h*	*j*	99.700665	95.620843	85.676275	**108.625277**	70.376940
249	*e*	*f*	*g*	*i*	*j*	100.243363	96.426991	86.758850	**107.876106**	68.694690
250	*e*	*f*	*h*	*i*	*j*	95.745413	90.733945	105.240826	**106.032110**	62.247706
251	*e*	*g*	*h*	*i*	*j*	98.058036	83.169643	**107.555804**	106.272321	64.944196
252	*f*	*g*	*h*	*i*	*j*	93.131991	83.355705	**107.539150**	106.767338	69.205817

Table 9.2.3.1 (part 5 of 5): links in example A53

9.3. Condorcet Criterion for Multi-Winner Elections

In this section, we will propose a generalization of the Condorcet criterion to multi-winner elections. The Condorcet criterion for single-winner elections (section 4.7) is important because, when there is a Condorcet winner $b \in A$, then it is still a Condorcet winner when alternatives $a_1,...,a_n \in A \setminus \{b\}$ are removed. So an alternative $b \in A$ doesn't owe his property of being a Condorcet winner to the presence of some other alternatives. Therefore, when we declare a Condorcet winner $b \in A$ elected whenever a Condorcet winner exists, we know that no other alternatives $a_1,...,a_n \in A \setminus \{b\}$ have changed the result of the election without being elected.

Therefore, a generalization of the Condorcet criterion to multi-winner elections should have the following properties:

- It should not be possible that there are more than M Condorcet winners (where M is the number of seats). This property is important because the Condorcet winners will later be declared the winners.

- Suppose $b \in A$ is a Condorcet winner. Then it should still be a Condorcet winner when alternatives $a_1,...,a_n \in A \setminus \{b\}$ are removed.

- The definition for “Condorcet winners” should be as weak as possible so that there are as many Condorcet winners as possible.

We propose the following generalization:

(9.3.1) In multi-winner elections, a *Condorcet winner* is an alternative $b \in A$ that wins in every (M+1)-way contest. Suppose $\mathcal{S}_M|_B$ (with $\emptyset \neq \mathcal{S}_M|_B \subseteq A_M$) is the set of potential winning sets when the used method to fill M seats is applied to the set B (with $\emptyset \neq B \subseteq A$ and $|\,B\,| > M$). Then we get:

(9.3.1a) $b \in A$ is a *Condorcet winner* : $\Leftrightarrow$

$$\forall\, \emptyset \neq B \subseteq A \text{ with } b \in B \text{ and } |\,B\,| = (M+1)\ \forall\, \mathcal{A} \in \mathcal{S}_M|_B\text{: } b \in \mathcal{A}.$$

The *Condorcet criterion* says that, when there is a Condorcet winner, then it should also be a winner overall. In short:

(9.3.1b) $b \in A$ is a Condorcet winner. $\Rightarrow$ ($\forall\, \mathcal{A} \in \mathcal{S}_M$: $b \in \mathcal{A}$.)

When $\succ_D$ satisfies (2.1.5) then for M = 1:

- (9.3.1a) is identical to (4.7.5) and (4.11.1.1).

- (9.3.1b) is identical to (4.7.6).

(9.3.2) In multi-winner elections, a *weak Condorcet winner* is an alternative $b \in A$ that wins or is tied for winning/losing in every (M+1)-way contest. In short:

(9.3.2a) $b \in A$ is a *weak Condorcet winner* : $\Leftrightarrow$

$$\forall\ \emptyset \neq B \subseteq A \text{ with } b \in B \text{ and } |\,B\,| = (M{+}1)\ \exists\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M|_B\text{: } b \in \underset{\sim}{\mathbb{A}}.$$

A weak Condorcet winner should win or be tied for winning/ losing overall. In short:

(9.3.2b) $b \in A$ is a weak Condorcet winner. $\Rightarrow$ ($\exists\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M$: $b \in \underset{\sim}{\mathbb{A}}$.)

When $\succ_D$ satisfies (2.1.4) and (2.1.5) then for $M = 1$:

- (9.3.2a) is identical to (4.11.1.2).
- (9.3.2b) is identical to (4.11.1.6).

(9.3.3) In multi-winner elections, A *Condorcet loser* is an alternative $b \in A$ that loses in every (M+1)-way contest. In short:

(9.3.3a) $b \in A$ is a *Condorcet loser* : $\Leftrightarrow$

$$\forall\ \emptyset \neq B \subseteq A \text{ with } b \in B \text{ and } |\,B\,| = (M{+}1)\ \forall\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M|_B\text{: } b \notin \underset{\sim}{\mathbb{A}}.$$

A Condorcet loser should be a loser overall. In short:

(9.3.3b) $b \in A$ is a Condorcet loser. $\Rightarrow$ ($\forall\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M$: $b \notin \underset{\sim}{\mathbb{A}}$.)

When $\succ_D$ satisfies (2.1.5) then for $M = 1$:

- (9.3.3a) is identical to (4.7.7) and (4.11.2.1).
- (9.3.3b) is identical to (4.7.8).

(9.3.4) In multi-winner elections, a *weak Condorcet loser* is an alternative $b \in A$ that loses or is tied for winning/losing in every (M+1)-way contest. In short:

(9.3.4a) $b \in A$ is a *weak Condorcet loser* : $\Leftrightarrow$

$$\forall\ \emptyset \neq B \subseteq A \text{ with } b \in B \text{ and } |\,B\,| = (M{+}1)\ \exists\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M|_B\text{: } b \notin \underset{\sim}{\mathbb{A}}.$$

A weak Condorcet loser should lose or be tied for winning/ losing overall. In short:

(9.3.4b) $b \in A$ is a weak Condorcet loser. $\Rightarrow$ ($\exists\ \underset{\sim}{\mathbb{A}} \in \mathcal{S}_M$: $b \notin \underset{\sim}{\mathbb{A}}$.)

When $\succ_D$ satisfies (2.1.4) and (2.1.5) then for $M = 1$:

- (9.3.4a) is identical to (4.11.2.2).
- (9.3.4b) is identical to (4.11.2.9).

It is important to keep in mind that, in multi-winner elections, the terms "Condorcet winner", "weak Condorcet winner", "Condorcet loser", and "weak Condorcet loser" always refer to the specific election method. For example, plurality-at-large will lead to different Condorcet winners than an STV method. So in multi-winner elections, the Condorcet criterion rather refers to the inner logic of the specific election method than to alternatives that must be elected regardless of the election method used.

If $\succ_{D2}$ satisfies (2.1.5), then we get for Schulze STV:

(9.3.5) $b \in A$ is a Condorcet winner $\Leftrightarrow$

$$\forall \{a_1,...,a_M\} \subseteq A \setminus \{b\} \ \exists \ a_i \in \{a_1,...,a_M\}: \\ N[\{a_1,...,a_M\};b] < N[(\{a_1,...,a_M,b\} \setminus \{a_i\});a_i].$$

(9.3.6) $b \in A$ is a weak Condorcet winner $\Leftrightarrow$

$$\forall \{a_1,...,a_M\} \subseteq A \setminus \{b\} \ \exists \ a_i \in \{a_1,...,a_M\}: \\ N[\{a_1,...,a_M\};b] \leq N[(\{a_1,...,a_M,b\} \setminus \{a_i\});a_i].$$

(9.3.7) $b \in A$ is a Condorcet loser $\Leftrightarrow$

$$\forall \{a_1,...,a_M\} \subseteq A \setminus \{b\} \ \forall \ a_i \in \{a_1,...,a_M\}: \\ N[\{a_1,...,a_M\};b] > N[(\{a_1,...,a_M,b\} \setminus \{a_i\});a_i].$$

(9.3.8) $b \in A$ is a weak Condorcet loser $\Leftrightarrow$

$$\forall \{a_1,...,a_M\} \subseteq A \setminus \{b\} \ \forall \ a_i \in \{a_1,...,a_M\}: \\ N[\{a_1,...,a_M\};b] \geq N[(\{a_1,...,a_M,b\} \setminus \{a_i\});a_i].$$

In example A53, the alternatives *a*, *g*, and *j* win in every 5-way contest; therefore, these alternatives should also win overall. The alternative *d* is tied for winning in one case (line 27) and wins in every other case; therefore, this alternative should win or be tied for winning/losing overall.

While there can be up to *M* Condorcet winners, there cannot be more than one Condorcet loser.

Claim:

If $\succ_{D2}$ satisfies (2.1.5), then Schulze STV, as defined in section 9.1, satisfies the Condorcet criterion for multi-winner electons, as defined in (9.3.1).

Proof:

Suppose alternative $b \in A$ is a Condorcet winner. Suppose $\{a_1,...,a_M\} \subseteq A \setminus \{b\}$.

We apply Schulze STV, as defined in section 9.1, on $\{a_1,...,a_M,b\}$. Suppose $c \in \{a_1,...,a_M,b\}$ is an alternative with maximum $N[(\{a_1,...,a_M,b\} \setminus \{c\});c]$. Then $(N[(\{a_1,...,a_M,b\} \setminus \{c\});c], N[(\{a_1,...,a_M\});b])$ is a win. With (9.3.5), we get that alternative c cannot be identical to alternative b. Therefore, the link $(\{a_1,...,a_M,b\} \setminus \{c\}) \rightarrow \{a_1,...,a_M\}$ is a path from $(\{a_1,...,a_M,b\} \setminus \{c\})$ to $\{a_1,...,a_M\}$ that contains only wins.

On the other side, there cannot be a path from $\{a_1,...,a_M\}$ to $(\{a_1,...,a_M,b\} \setminus \{c\})$ that contains only wins because any path from $\{a_1,...,a_M\}$ to $(\{a_1,...,a_M,b\} \setminus \{c\})$ must contain a link from a set $\mathfrak{C}(i)$ with $b \notin \mathfrak{C}(i)$ to a set $\mathfrak{C}(i+1)$ with $b \in \mathfrak{C}(i+1)$. But because of the definition of Condorcet winners, the link $\mathfrak{C}(i) \rightarrow \mathfrak{C}(i+1)$ must be a tie or a defeat.

With (2.1.5), we get that every path that contains only wins is stronger than every path that contains a tie or a defeat.

Therefore, every set $\{a_1,...,a_M\}$, that does not contain alternative b, is disqualified by some set that contains alternative b. □

The proofs that Schulze STV satisfies (9.3.2b), (9.3.2c), and (9.3.2d) are analogue to the proofs for (4.11.1.6), (4.11.2.9), and (9.3.2a).

In a similar manner, we can generalize the Smith criterion (section 4.7) to multi-winner elections.

Definition:

A multi-winner election method, where M is the number of seats, satisfies the *Smith criterion for multi-winner elections*, if the following holds:

Suppose $\emptyset \neq B \subsetneq A$. Suppose $x \in \mathbb{N}$ with $1 \leq x \leq |B|$ and $x \leq M$.

(9.3.9) Suppose, for every $y \in \mathbb{N}$ with $1 \leq y \leq x$, we have: In every $(M+1)$-contest between y alternatives of the set B and $M+1-y$ alternatives of $A \setminus B$ each of the alternatives of the set B is in every potential winning set.

Then every potential winning set contains at least x alternatives of the set B.

In short, a multi-winner election method, where M is the number of seats, satisfies the *Smith criterion for multi-winner elections*, if the following holds:

$$\forall\, \emptyset \neq B \subsetneq A\ \forall\, x \in \mathbb{N} \text{ with } 1 \leq x \leq |B| \text{ and } x \leq M\text{:}$$

$$(\,(\,\forall\, y \in \mathbb{N} \text{ with } 1 \leq y \leq x$$

$$\forall\, \emptyset \neq \tilde{A} \subseteq A \text{ with } |\tilde{A}| = (M+1) \text{ and } |\tilde{A} \cap B| = y$$

$$\forall\, \mathfrak{A} \in \mathcal{S}_M|_{\tilde{A}}\text{: } |\mathfrak{A} \cap B| = y.\,)$$

$$\Rightarrow (\,\forall\, \mathfrak{A} \in \mathcal{S}_M\text{: } |\mathfrak{A} \cap B| \geq x.\,)\,)$$

Example:

> There are $C = 4$ alternatives running for $M = 2$ seats.
> When $\{a,b,c\}$ are running, the unique winning set is $\{a,b\}$.
> When $\{a,b,d\}$ are running, the unique winning set is $\{a,b\}$.
> When $\{a,c,d\}$ are running, the unique winning set is $\{c,d\}$.
> When $\{b,c,d\}$ are running, the unique winning set is $\{c,d\}$.

In the example above, alternatives a and b are winners whenever they and exactly one other alternative are running. Furthermore, alternatives c and d are winners whenever they and exactly one other alternative are running. The Smith criterion for multi-winner elections doesn’t say anything in the example above. This shows that the Smith criterion for multi-winner elections demands more than just local stability.

Claim:

If $\succ_{D2}$ satisfies (2.1.5), then Schulze STV, as defined in section 9.1, satisfies the Smith criterion for multi-winner electons.

Proof (overview):

The proof that Schulze STV satisfies the Smith criterion for multi-winner elections is analogous to the proof that Schulze STV satisfies the Condorcet criterion for multi-winner elections.

Part 1: Suppose $z \in \mathbb{N}_0$ with $0 \leq z < x$. Suppose $\{a_1,...,a_{(M-z)},b_1,...,b_z\}$ is a set of $M-z$ alternatives $a_1,...,a_{(M-z)} \in A \setminus B$ and z alternatives $b_1,...,b_z \in B$. Suppose $b_{(z+1)} \in B \setminus \{b_1,...,b_z\}$ is an arbitrarily chosen alternative. Suppose $c \in \{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\}$ is an alternative with maximum $N[(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\});c]$. Then $(N[(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\});c],\ N[(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\});b_{(z+1)}])$ is a win. With (9.3.9), we get $c \notin \{b_1,...,b_z,b_{(z+1)}\}$. Therefore, the link $(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\}) \rightarrow \{a_1,...,a_{(M-z)},b_1,...,b_z\}$ is a path from $(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\})$ to $\{a_1,...,a_{(M-z)},b_1,...,b_z\}$ that contains only wins.

On the other side, there cannot be a path from $\{a_1,...,a_{(M-z)},b_1,...,b_z\}$ to $(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\})$ that contains only wins because any path from $\{a_1,...,a_{(M-z)},b_1,...,b_z\}$ to $(\{a_1,...,a_{(M-z)},b_1,...,b_z,b_{(z+1)}\} \setminus \{c\})$ must contain a link from a set $\mathfrak{C}(i)$ with z alternatives from the set B to a set $\mathfrak{C}(i+1)$ with $z+1$ alternatives from the set B. But with (9.3.9), we get that the link $\mathfrak{C}(i) \rightarrow \mathfrak{C}(i+1)$ must be a tie or a defeat.

With (2.1.5), we get that every path that contains only wins is stronger than every path that contains a tie or a defeat.

Therefore, every set $\{a_1,...,a_{(M-z)},b_1,...,b_z\}$, that contains only z alternatives from the set B is disqualified by some set that contains $z+1$ alternatives from the set B.

Part 2: Part 1 is applied to $z := 0,...,(x-1)$. As indirect defeats are transitive (section 4.1), we get that every set with less than x alternatives from the set B is disqualified by some set with x alternatives from the set B.□

The Smith criterion for multi-winner elections implies the Condorcet criterion for multi-winner elections. We get the Condorcet criterion for multi-winner elections when we restrict the Smith criterion for multi-winner elections to sets with exactly one alternative.

In example A53, the Smith criterion for multi-winner elections implies that at least one winner must come from the set $\{d,f\}$ because, whenever exactly one alternative from the set $\{d,f\}$ and exactly four alternatives from $A \setminus \{d,f\}$ are running, the alternative from $\{d,f\}$ is a winner of Schulze STV.

9.4. Proportionality

Definition (Dummett-Droop Proportionality):

A preferential multi-winner election method satisfies *Dummett-Droop proportionality* (DDP) if the following holds for every $\emptyset \neq B \subsetneq A$ and for every $x \in \mathbb{N}$ with $x \leq |\,B\,|$:

Suppose that strictly more than $x \cdot N \,/\, (M+1)$ voters strictly prefer every alternative in B to every other alternative. In other words:

$$\|\{\, v \in V \mid \forall\, a \in B\ \forall\, b \notin B\colon a \succ_v b \,\}\| > x \cdot N \,/\, (M+1).$$

Then at least x alternatives of set B must be elected.

It has been proposed by Droop (1881) that an alternative should be elected as soon as it has received more than $N \,/\, (M+1)$ votes. This idea has been generalized by Dummett (1984) to sets of alternatives. Today, DDP is considered a necessary and sufficient criterion for every preferential multi-winner election method to qualify as an STV method.

Claim:

Schulze STV, as defined in section 9.1, satisfies Dummett-Droop proportionality.

Proof (overview):

The proof is ommitted because it is similar to the proof that Schulze STV satisfies the Smith criterion for multi-winner elections (section 9.3). □

10. Proportional Ranking

When proportional representation by party lists is being used, then each party has to submit in advance a linear order of its candidates without knowing how many seats it will win. Frequently, the parties are interested that — however many candidates are elected — the elected candidates reflect the strengths of the different party wings in a manner as proportional as possible (Otten, 1998, 2000; Rosenstiel, 1998; Warren, 1999; Skowron, 2017). We will call a linear order with this property a *proportional ranking*. The two most important approaches to produce a proportional ranking are the *bottom-up* approach (Rosenstiel, 1998) and the *top-down* approach (Otten, 1998, 2000).

The *bottom-up* approach says that we start with the situation where all C candidates are elected. Then, for $k = C$ to 2, we ask which candidate can be eliminated (without changing who is already eliminated) so that the distortion of the proportionality of the remaining candidates is as small as possible; the newly eliminated candidate then gets the k-th place of this party list.

The *top-down* approach says that we use a single-winner election method to fill the first place of this party list. Then, for $k = 2$ to C, we ask which candidate can be added to the already elected candidates (without changing who is already elected) so that the distortion of the proportionality is as small as possible; the newly added candidate then gets the k-th place of this party list.

I prefer the top-down approach to the bottom-up approach, because the bottom-up approach starts with the lowest and, therefore, (as the number of candidates is usually significantly larger than the number of seats this party can realistically hope to win) least important places so that slight fluctuations in the filling of the lowest places can have an enormous impact on the filling of the best places. Therefore, in this paper we presume that the top-down approach is being used.

In section 10.1, we will propose a new proportional ranking method. In sections 10.2 and 10.3, we will apply this method to the examples of Tideman’s database. The proposed proportional ranking method is based on the following idea:

- Suppose $a_1,...,a_{(k-1)} \in A$ are already elected.
- Suppose there are candidates $\emptyset \neq \{b_1,...,b_z\} \subseteq A \setminus \{a_1,...,a_{(k-1)}\}$ such that, whenever some candidate $b_j \in \{b_1,...,b_z\}$ is added to $\{a_1,...,a_{(k-1)}\}$, then choosing the set $\{a_1,...,a_{(k-1)},b_j\}$ is compatible to the Smith criterion for k-winner elections (section 9.3).
- Then the k-th seat should go to one of the candidates in $\{b_1,...,b_z\}$.

10.1. Schulze Proportional Ranking

Proportional completion is defined in section 9.1.1.
$N[\{a_1,...,a_k\};b]$ is defined in section 9.1.2.
$\succ_{D1}$ and $\succ_{D2}$ are two binary relations that each satisfy (2.1.1) – (2.1.3).

Stage 1:

We calculate the Schulze single-winner ranking $\mathcal{O}_1$ on A, as defined in section 5, with $\succ_{D1}$.

Stage 2:

Proportional completion is used to complete V to W.

Stage 3:

For k : = 1 to (C–1) do
{
Suppose $a_1,...,a_{(k-1)}$ are already elected.

For each pair of alternatives $b,c \notin \{a_1,...,a_{(k-1)}\}$, we define:

$$H[b,c] := N[\{a_1,...,a_{(k-1)},b\};c].$$

We apply the Schulze single-winner election method, as defined in section 2.3 stage 2, on $H[i,j]$, instead of $N[i,j]$, and with $\succ_{D2}$. If there is only one potential winner, then it gets the k-th place. If there is more than one potential winner, then the k-th place goes to that potential winner b with $bc \in \mathcal{O}_1$ for every other potential winner c.
}

10.2. Example A53

The following series of tables illustrates the Schulze proportional ranking method when applied to example A53 of Tideman’s database. Pairwise wins are **fat and underlined**. Pairwise ties are *italic and underlined*.

	N[*;*a*]	*N*[*;*b*]	*N*[*;*c*]	*N*[*;*d*]	*N*[*;*e*]	*N*[*;*f*]	*N*[*;*g*]	*N*[*;*h*]	*N*[*;*i*]	*N*[*;*j*]
N[*a*;*]	--	**316.175711**	**352.129380**	**303.100775**	**307.846154**	**298.883249**	**266.374696**	**349.351351**	**348.337731**	193.625304
N[*b*;*]	143.824289	--	**262.462462**	221.153846	**240.176991**	222.913165	199.414894	**265.438066**	**263.253012**	128.845209
N[*c*;*]	107.870620	197.537538	--	197.906977	201.703470	183.197674	171.397260	**240.747664**	**241.022364**	105.891089
N[*d*;*]	156.899225	**238.846154**	**262.093023**	--	**248.295455**	**242.234043**	197.142857	**264.532578**	**276.260623**	146.975610
N[*e*;*]	152.153846	219.823009	**258.296530**	211.704545	--	214.494382	191.152815	**259.814815**	**274.842767**	120.992556
N[*f*;*]	161.116751	**237.086835**	**276.802326**	217.765957	**245.505618**	--	207.817259	**280.229885**	**275.190616**	139.803922
N[*g*;*]	193.625304	**260.585106**	**288.602740**	**262.857143**	**268.847185**	**252.182741**	--	**306.259947**	**314.604905**	183.785047
N[*h*;*]	110.648649	194.561934	219.252336	195.467422	200.185185	179.770115	153.740053	--	**250.125000**	87.058824
N[*i*;*]	111.662269	196.746988	218.977636	183.739377	185.157233	184.809384	145.395095	209.875000	--	97.150127
N[*j*;*]	**266.374696**	**331.154791**	**354.108911**	**313.024390**	**339.007444**	**320.196078**	**276.214953**	**372.941176**	**362.849873**	--

The 1. place goes to alternative *j*.

	N[{*j*,*};*a*]	*N*[{*j*,*};*b*]	*N*[{*j*,*};*c*]	*N*[{*j*,*};*d*]	*N*[{*j*,*};*e*]	*N*[{*j*,*};*f*]	*N*[{*j*,*};*g*]	*N*[{*j*,*};*h*]	*N*[{*j*,*};*i*]
N[{*a*,*j*};*]	--	**188.909513**	**204.084507**	**184.640371**	**188.568129**	**185.604651**	**164.582393**	**208.844340**	**201.113744**
N[{*b*,*j*};*]	143.824289	--	**193.995327**	171.444954	**185.831382**	174.389671	155.588235	**200.287081**	**196.515513**
N[{*c*,*j*};*]	107.870620	178.948598	--	172.097902	181.492891	173.849765	153.507973	**201.318945**	**192.673031**
N[{*d*,*j*};*]	156.357309	**183.050459**	**193.006993**	--	**187.645688**	**179.186047**	156.628959	**196.988235**	**197.605634**
N[{*e*,*j*};*]	149.792148	179.367681	**195.118483**	172.097902	--	175.754717	157.313770	**201.802885**	**200.555556**
N[{*f*,*j*};*]	148.162791	**180.328638**	**192.206573**	171.162791	**182.264151**	--	157.123596	**200.428571**	**198.162291**
N[{*g*,*j*};*]	160.948081	**188.891403**	**200.136674**	**184.208145**	**191.060948**	**182.449438**	--	**207.159353**	**204.210526**
N[{*h*,*j*};*]	110.648649	173.325359	183.669065	165.058824	175.264423	165.928571	144.480370	--	191.477833
N[{*i*,*j*};*]	111.662269	176.754177	187.732697	165.751174	178.333333	169.069212	145.395095	**194.876847**	--

The 2. place goes to alternative *a*.

	N[{*a*,*j*,*};*b*]	*N*[{*a*,*j*,*};*c*]	*N*[{*a*,*j*,*};*d*]	*N*[{*a*,*j*,*};*e*]	*N*[{*a*,*j*,*};*f*]	*N*[{*a*,*j*,*};*g*]	*N*[{*a*,*j*,*};*h*]	*N*[{*a*,*j*,*};*i*]
N[{*a*,*b*,*j*};*]	--	**139.742424**	126.860987	**132.267267**	127.603930	115.511111	**142.105263**	**137.929985**
N[{*a*,*c*,*j*};*]	130.333333	--	126.560847	129.287879	127.308869	112.604167	**141.376147**	137.081413
N[{*a*,*d*,*j*};*]	**131.674141**	**139.077853**	--	**131.674141**	**129.342404**	116.356932	**141.511716**	**138.348485**
N[{*a*,*e*,*j*};*]	129.504505	**141.136364**	127.892377	--	128.702866	117.035398	**142.554800**	**141.430746**
N[{*a*,*f*,*j*};*]	**130.733182**	**140.321101**	125.865457	**130.090498**	--	115.851852	**143.204252**	**140.000000**
N[{*a*,*g*,*j*};*]	**133.911111**	**144.092262**	**133.318584**	**135.014749**	**131.866667**	--	**145.769806**	**143.685393**
N[{*a*,*h*,*j*};*]	128.771930	137.859327	124.822373	127.603930	124.692483	110.702541	--	137.324053
N[{*a*,*i*,*j*};*]	129.878234	**138.847926**	126.151515	129.178082	126.666667	113.707865	**140.525909**	--

The 3. place goes to alternative *g*.

	N[{*a*,*g*,*j*,*};*b*]	*N*[{*a*,*g*,*j*,*};*c*]	*N*[{*a*,*g*,*j*,*};*d*]	*N*[{*a*,*g*,*j*,*};*e*]	*N*[{*a*,*g*,*j*,*};*f*]	*N*[{*a*,*g*,*j*,*};*h*]	*N*[{*a*,*g*,*j*,*};*i*]
N[{*a*,*b*,*g*,*j*};*]	--	**108.667401**	101.411379	**104.131868**	101.068282	**110.410200**	**109.122222**
N[{*a*,*c*,*g*,*j*};*]	102.334802	--	101.574890	102.615385	100.465632	**109.888889**	108.355556
N[{*a*,*d*,*g*,*j*};*]	**102.166302**	**108.414097**	--	**102.921225**	**101.351648**	**110.197802**	**108.907285**
N[{*a*,*e*,*g*,*j*};*]	101.351648	**108.934066**	101.663020	--	101.321586	**110.674779**	**109.402655**
N[{*a*,*f*,*g*,*j*};*]	**102.334802**	**109.135255**	101.098901	**102.841410**	--	**110.665188**	**109.390244**
N[{*a*,*g*,*h*,*j*};*]	101.230599	108.611111	100.087912	101.769912	99.190687	--	108.568233
N[{*a*,*g*,*i*,*j*};*]	101.966667	**108.866667**	101.291391	102.533186	99.955654	**109.597315**	--

The 4. place goes to alternative *d*.

	N[{*a*,*d*,*g*,*j*,*};*b*]	*N*[{*a*,*d*,*g*,*j*,*};*c*]	*N*[{*a*,*d*,*g*,*j*,*};*e*]	*N*[{*a*,*d*,*g*,*j*,*};*f*]	*N*[{*a*,*d*,*g*,*j*,*};*h*]	*N*[{*a*,*d*,*g*,*j*,*};*i*]
N[{*a*,*b*,*d*,*g*,*j*};*]	--	**87.189542**	**84.383442**	*82.579521*	**88.986900**	**88.175055**
N[{*a*,*c*,*d*,*g*,*j*};*]	82.579521	--	83.362445	81.687912	**88.570175**	**87.551648**
N[{*a*,*d*,*e*,*g*,*j*};*]	82.178649	**87.379913**	--	82.358079	**89.181619**	**88.175055**
N[{*a*,*d*,*f*,*g*,*j*};*]	*82.579521*	**87.551648**	**83.362445**	--	**88.973684**	**88.166667**
N[{*a*,*d*,*g*,*h*,*j*};*]	82.157205	87.157895	82.739606	81.307018	--	87.753846
N[{*a*,*d*,*g*,*i*,*j*};*]	82.135667	87.349451	83.142232	81.508772	**88.158242**	--

The 5. place goes to alternative *f* (because alternative *f* is ranked above alternative *b* in the single-winner ranking; i.e. $fb \in O_1$).

	N[{*a*,*d*,*f*,*g*,*j*,*};*b*]	*N*[{*a*,*d*,*f*,*g*,*j*,*};*c*]	*N*[{*a*,*d*,*f*,*g*,*j*,*};*e*]	*N*[{*a*,*d*,*f*,*g*,*j*,*};*h*]	*N*[{*a*,*d*,*f*,*g*,*j*,*};*i*]
N[{*a*,*b*,*d*,*f*,*g*,*j*};*]	--	**73.326071**	**71.000000**	**74.662309**	**73.994190**
N[{*a*,*c*,*d*,*f*,*g*,*j*};*]	69.484386	--	70.138282	**74.144737**	**73.640351**
N[{*a*,*d*,*e*,*f*,*g*,*j*};*]	69.166667	**73.151383**	--	**74.657933**	**73.988355**
N[{*a*,*d*,*f*,*g*,*h*,*j*};*]	69.150327	73.304094	69.803493	--	73.976608
N[{*a*,*d*,*f*,*g*,*i*,*j*};*]	69.150327	73.304094	70.138282	**74.144737**	--

The 6. place goes to alternative *b*.

	N[{*a*,*b*,*d*,*f*,*g*,*j*,*};*c*]	*N*[{*a*,*b*,*d*,*f*,*g*,*j*,*};*e*]	*N*[{*a*,*b*,*d*,*f*,*g*,*j*,*};*h*]	*N*[{*a*,*b*,*d*,*f*,*g*,*j*,*};*i*]
N[{*a*,*b*,*c*,*d*,*f*,*g*,*j*};*]	--	61.285714	**63.996265**	**63.566760**
N[{*a*,*b*,*d*,*e*,*f*,*g*,*j*};*]	**63.000000**	--	**64.000000**	**63.571429**
N[{*a*,*b*,*d*,*f*,*g*,*h*,*j*};*]	63.137255	61.000000	--	63.709928
N[{*a*,*b*,*d*,*f*,*g*,*i*,*j*};*]	63.137255	61.285714	**63.996265**	--

The 7. place goes to alternative *e*.

	N[{*a*,*b*,*d*,*e*,*f*,*g*,*j*,*};*c*]	*N*[{*a*,*b*,*d*,*e*,*f*,*g*,*j*,*};*h*]	*N*[{*a*,*b*,*d*,*e*,*f*,*g*,*j*,*};*i*]
N[{*a*,*b*,*c*,*d*,*e*,*f*,*g*,*j*};*]	--	**56.000000**	**55.750000**
N[{*a*,*b*,*d*,*e*,*f*,*g*,*h*,*j*};*]	55.375000	--	55.750000
N[{*a*,*b*,*d*,*e*,*f*,*g*,*i*,*j*};*]	55.375000	**56.000000**	--

The 8. place goes to alternative *c*.

	N[{*a*,*b*,*c*,*d*,*e*,*f*,*g*,*j*,*};*h*]	*N*[{*a*,*b*,*c*,*d*,*e*,*f*,*g*,*j*,*};*i*]
N[{*a*,*b*,*c*,*d*,*e*,*f*,*g*,*h*,*j*};*]	--	49.555556
N[{*a*,*b*,*c*,*d*,*e*,*f*,*g*,*i*,*j*};*]	**49.777778**	--

The 9. place goes to alternative *i*.

The 10. place goes to alternative *h*.

So, the Schulze proportional ranking is *j a g d f b e c i h*.

10.3. Tideman’s Database

In table 10.3.1, Schulze STV and Schulze proportional ranking are applied to the instances of Tideman’s (2000) database. We use $\succ_{ratio}$ for $\succ_{D1}$ because $\succ_{ratio}$ corresponds to proportional completion; the fact that we use $\succ_{ratio}$ for $\succ_{D1}$ means that it makes no difference whether we first calculate the Schulze single-winner ranking O_1 and then apply proportional completion or first apply proportional completion and then calculate the Schulze single-winner ranking O_1. We use $\succ_{margin}$ for $\succ_{D2}$ because of simplicity.

The column “name #1” contains the name of the instance. The column “name #2” contains the name of the same instance in Wichmann’s (1994) database. N is the number of voters. C is the number of alternatives. M is the number of seats.

Column “Dummett” contains the constraints given by “Dummett-Droop proportionality” (DDP), as defined in section 9.4. The constraints are separated by spaces. If this constraint consists of a single alternative, then this means that this alternative must be elected according to DDP. If this constraint has the form “*abcdef*(*i*)” then this means that at least *i* alternatives of the set $\{a,b,c,d,e,f\}$ must be elected according to DDP. For example, in example A35 the constraints are “*f eijkq*(1)” so that (1) alternative *f* must be elected and (2) at least one alternative of the set $\{e,i,j,k,q\}$ must be elected according to DDP. In 3 instances (A64, A72, A83), there is only one set of M alternatives that can be elected according to DDP.

The column “Condorcet winners” contains the Condorcet winners in Schulze STV [according to (9.3.5)]; alternatives, that are only weak Condorcet winners [according to (9.3.6)], are listed in brackets ( ). The column “Condorcet losers” contains the Condorcet losers in Schulze STV [according to (9.3.7)]; alternatives, that are only weak Condorcet losers [according to (9.3.8)], are listed in brackets ( ). It is important to keep in mind that, as long as $\succ_{D2}$ satisfies (2.1.5), the Condorcet winners in Schulze STV, the Condorcet losers in Schulze STV, and the possible winning sets according to the Smith criterion (for multi-winner elections) in Schulze STV do not depend on the specific choice for $\succ_{D2}$. As long as $\succ_{D2}$ satisfies (2.1.4) and (2.1.5), the weak Condorcet winners in Schulze STV and the weak Condorcet losers in Schulze STV do not depend on the specific choice for $\succ_{D2}$.

The column “Schulze STV” contains the winning set of Schulze STV, as defined in section 9.1.3. When several sets are tied for winning, then (rather than listing all potential winning sets) the winning set chosen by the tie-breaker, as defined in section 9.1.3 stage 4, is listed. In 3 instances (A34, A88, A97), an alternative, that is a weak Condorcet winner, is not elected. In instances A34 and A97, this is due to the fact that the number of alternatives, that are weak Condorcet winners or non-weak Condorcet winners, is larger than the number of seats. In instance A34, the sets $\{a,b,c,d,e,f,g,h,j,k,m,n\}$, $\{a,b,c,d,e,f,h,j,k,l,m,n\}$, $\{a,b,c,d,e,g,h,j,k,l,m,n\}$, $\{a,b,c,e,f,g,h,j,k,l,m,n\}$, and $\{b,c,d,e,f,g,h,j,k,l,m,n\}$ are tied for winning; the tie-breaker chooses $\{a,b,c,d,e,f,h,j,k,l,m,n\}$, because (1) $da \in \mathcal{O}_1$, $dl \in \mathcal{O}_1$, $df \in \mathcal{O}_1$, and $dg \in \mathcal{O}_1$ so that the set $\{a,b,c,e,f,g,h,j,k,l,m,n\}$ is disqualified at the first stage for not containing alternative d, (2) $al \in \mathcal{O}_1$, $af \in \mathcal{O}_1$, and $ag \in \mathcal{O}_1$ so that the set $\{b,c,d,e,f,g,h,j,k,l,m,n\}$ is disqualified at the second stage for not containing alternative a, (3) $lf \in \mathcal{O}_1$ and $lg \in \mathcal{O}_1$ so that the set $\{a,b,c,d,e,f,g,h,j,k,m,n\}$ is disqualified at the third stage for not containing alternative l, and (4) $fg \in \mathcal{O}_1$ so that the set $\{a,b,c,d,e,g,h,j,k,l,m,n\}$ is disqualified at the fourth stage for not containing alternative f. In instance A88, the sets $\{a,c,e,f,g,h\}$, $\{b,c,e,f,g,h\}$, and $\{c,d,e,f,g,h\}$ are tied for winning; while only alternative d is a weak Condorcet winner, the tie-breaker chooses $\{a,c,e,f,g,h\}$, because $ab \in \mathcal{O}_1$ and $ad \in \mathcal{O}_1$. In instance A97, the sets $\{a,b\}$ and $\{a,c\}$ are tied for winning; the tie-breaker chooses $\{a,b\}$, because $bc \in \mathcal{O}_1$.

The column “Schulze proportional ranking” contains the Schulze proportional ranking, as defined in section 10.1. When several rankings are tied for winning, then (rather than listing all potential rankings) the ranking chosen by the tie-breaker, as defined in section 10.1 stage 3 last sentence, is listed. In 4 instances (A10, A12, A33, A67), the Schulze proportional ranking is not unique even with the proposed tie-breaker. This is due to the fact that, in these instances, even the Schulze single-winner ranking $\mathcal{O}_1$ is not unique. Only in 6 of the 66 instances of Tideman’s database (A10, A11, A13, A33, A34, A59), the winning set of Schulze STV differs from the first M alternatives of Schulze proportional ranking.

The column “runtime” contains the runtime to calculate the Schulze STV winners. A Fujitsu RX 350S8 with two 6-core “E5-2630v2 @ 2.60 GHz” processors was used for the calculations. Hyper-threading was disabled. The programs to calculate the STV winners and the Schulze proportional ranking were written in Microsoft Visual C++ 2010.

	name #1	name #2	*N*	*C*	*M*	Dummett	Condorcet winners	Condorcet losers	Schulze STV	Schulze proportional ranking	runtime
1	A01	R006	380	10	3	*a*	*a h i*	---	*a h i*	*a i h d b c g j f e*	< 0.1 s
2	A02	R007	371	9	2	---	*c d*	*g*	*c d*	*c d e b f a h i g*	< 0.1 s
3	A03	R008	989	15	7	*d f h*	*b d e f h k*	---	*b d e f h k n*	*f h d k b e n g a l c i j o m*	5.3 s
4	A04	R009	43	14	2	---	*i*	*d*	*f i*	*i f e a k c b g d h m j l n*	< 0.1 s
5	A05	R010	762	16	7	*a*	*a c d e g l m*	---	*a c d e g l m*	*a c m e d g l k f o p h i j b n*	7.4 s
6	A06	R011	280	9	5	*i*	*c e h i*	---	*b c e h i*	*i h e c b f g a d*	< 0.1 s
7	A07	R012	79	17	2	---	(*d*) *i*	*f*	*d i*	*i d c o m p h a k g e j l n f b q*	< 0.1 s
8	A08	R013	78	7	2	*d*	*d g*	(*a*)	*d g*	*d g c b f e a*	< 0.1 s
9	A10	R015	83	19	3	---	*m n p*	---	*m n p*	*n* ((*a p m q*) or (*m p q a*)) *g f s r l i b d j k e h o c*	< 0.1 s
10	A11	R016	963	10	6	*a c*	*a c* (*e*) *h*	---	*a c d e g h*	*a c e h j g d i b f*	< 0.1 s
11	A12	R017	76	20	2	---	*i r*	---	*i r*	*r i l e g s a m p b h t n o k d* ((*f j*) or (*j f*)) *c q*	< 0.1 s
12	A13	R018	104	26	2	---	*t*	---	*k t*	*i t k m s j c f y z l u n a g e b p r d h v x o q w*	< 0.1 s
13	A14	R019	73	17	2	---	*b j*	---	*b j*	*j b c n h q o i a l e d g k p m f*	< 0.1 s
14	A15	R020	77	21	2	---	(*g*) *l*	---	*g l*	*l g t r m i c h p k j q s a b o d u n f e*	< 0.1 s
15	A17	R022	867	13	8	*a b j*	*a b d e f j l*	---	*a b d e f i j l*	*j b a e l f d i m h k c g*	0.5 s
16	A18	R023	976	6	4	*b c*	*a b c f*	*e*	*a b c f*	*b c f a d e*	< 0.1 s
17	A19	R024	860	7	3	---	*a e g*	*f*	*a e g*	*e a g c d b f*	< 0.1 s
18	A20	R025	2785	5	4	*a d*	*a c d e*	*b*	*a c d e*	*a d c e b*	< 0.1 s
19	A22	R027	44	11	2	*ck*(1)	(*c*) *k*	*f*	*c k*	*k c a g b d i j h e f*	< 0.1 s
20	A23	R028	91	29	2	---	3 5	---	3 5	3-5-21-7-27-26-22-9-17-14-15-24-4-16-19-20-6-11-18-28-2-23-29-1-13-8-10-12-25	< 0.1 s

Table 10.3.1 (part 1 of 3): Schulze STV applied to instances of Tideman’s database

	name #1	name #2	N	C	M	Dummett	Condorcet winners	Condorcet losers	Schulze STV	Schulze proportional ranking	runtime
21	A33	R038	9	18	3	---	(o)	(j)	e o q	o a e i h c l n q f r d g ((b m p) or (b p m) or (m b p)) k j	< 0.1 s
22	A34	R039	63	14	12	b e h j n	(a) b c (d) e (f) (g) h j k (l) m n	(i)	a b c d e f h j k l m n	j b h e k n l g m c d a f i	< 0.1 s
23	A35	R040	176	17	5	f eijkq(1)	a (d) e f	---	a d e f q	f e a q d k b i m n c h j p o g l	2.9 s
24	A48	R041	923	10	9	b c d e f	a b c d e f g h j	i	a b c d e f g h j	d f b e c h j g a i	< 0.1 s
25	A49	R042	575	13	3	h	a c h	k	a c h	h c a j l d m g b i e f k	< 0.1 s
26	A51	R044	42	6	3	d	a d e	b	a d e	d a e f c b	< 0.1 s
27	A52	R045	667	10	6	d e	a b c d e g	h	a b c d e g	e d b g a c j f i h	< 0.1 s
28	A53	R046	460	10	4	j	a (d) g j	---	a d g j	j a g d f b e c i h	< 0.1 s
29	A54	R047	924	11	9	a d e f k	a b d e f g h j k	---	a b d e f g h j k	e d f a k g h j b i c	< 0.1 s
30	A55	R048	302	10	5	i	a (d) f i j	b	a d f i j	i a j f d e h c g b	< 0.1 s
31	A56	R049	685	13	2	---	j k	---	j k	j k f h m g d a e c b l i	< 0.1 s
32	A57	R050	310	9	2	de(1)	d e	---	d e	d e i b h c g f a	< 0.1 s
33	A59	R052	694	7	4	d f	d f g	---	b d f g	f d e g b c a	< 0.1 s
34	A63	R056	156	7	2	---	c f	---	c f	c f e d b a g	< 0.1 s
35	A64	R057	196	3	2	b c	b c	a	b c	b c a	< 0.1 s
36	A65	R058	198	10	6	b g	b e f g j	---	a b e f g j	g b f e j a d c h i	< 0.1 s
37	A66	R059	193	6	4	f	b d e f	a	b d e f	f d e b c a	< 0.1 s
38	A67	R060	183	14	10	b f g k	b c e f g i j k l	---	b c e f g h i j k l	((f g) or (g f)) k b i e j l c h n m d a	4.0 s
39	A68	R061	50	4	3	a c	a c d	b	a c d	a c d b	< 0.1 s
40	A69	R062	86	9	3	---	a c e	---	a c e	e c a f i d b h g	< 0.1 s

Table 10.3.1 (part 2 of 3): Schulze STV applied to instances of Tideman's database

	name #1	name #2	*N*	*C*	*M*	Dummett	Condorcet winners	Condorcet losers	Schulze STV	Schulze proportional ranking	runtime
41	A70	R063	529	9	3	*e*	*e h i*	---	*e h i*	*e i h c d b a g f*	< 0.1 s
42	A71	R064	500	8	7	*d g*	*a b c d e f g*	*h*	*a b c d e f g*	*d c g e a b f h*	< 0.1 s
43	A72	R065	272	3	2	*a c*	*a c*	*b*	*a c*	*a c b*	< 0.1 s
44	A73	R066	525	5	2	---	*c d*	---	*c d*	*d c b a e*	< 0.1 s
45	A74	R067	253	3	2	*a*	*a c*	*b*	*a c*	*a c b*	< 0.1 s
46	A76	R069	403	5	2	*c*	*a c*	---	*a c*	*c a d b e*	< 0.1 s
47	A78	R071	486	4	3	*c d*	*b c d*	*a*	*b c d*	*c d b a*	< 0.1 s
48	A79	R072	362	8	4	*g*	*a c e g*	---	*a c e g*	*g a e c f d b h*	< 0.1 s
49	A80	R073	269	7	5	*a*	*a b c e g*	---	*a b c e g*	*a e c g b f d*	< 0.1 s
50	A81	R074	902	11	9	*b c e h j*	*a b c e g h i j k*	*f*	*a b c e g h i j k*	*h e c b j g a i k d f*	< 0.1 s
51	A83	R076	1123	4	3	*a b c*	*a b c*	*d*	*a b c*	*c a b d*	< 0.1 s
52	A84	R077	277	7	6	*b c e*	*a b c d e g*	*f*	*a b c d e g*	*e b c d g a f*	< 0.1 s
53	A85	R078	158	4	3	*a d*	*a b d*	*c*	*a b d*	*d a b c*	< 0.1 s
54	A86	R079	157	5	4	*c*	*a c d e*	*b*	*a c d e*	*c a d e b*	< 0.1 s
55	A87	R080	120	4	3	*b d*	*a b d*	*c*	*a b d*	*d b a c*	< 0.1 s
56	A88	R081	135	9	6	*e h*	*c* (*d*) *e f g h*	---	*a c e f g h*	*h e g c f a d b i*	< 0.1 s
57	A89	R082	256	5	3	*e acd*(1)	*a d e*	*c*	*a d e*	*e d a b c*	< 0.1 s
58	A90	R083	366	20	12	---	*a* (*b*) *c d e f i l* (*n*) (*o*) *t*	---	*a b c d e f i l n o s t*	*a i t l e c s d f n o b p j m g k r h q*	49.0 s
59	A92	R085	540	13	3	*d*	*d f i*	---	*d f i*	*d f i e b h a m c j g k l*	< 0.1 s
60	A93	R086	561	4	2	---	*b d*	*a*	*b d*	*b d c a*	< 0.1 s
61	A94	R087	579	4	2	---	*a d*	*b*	*a d*	*a d b c*	< 0.1 s
62	A95	R088	587	7	2	---	*a* (*b*)	*c*	*a b*	*a b f g d e c*	< 0.1 s
63	A96	R089	564	6	2	---	*a b*	*c*	*a b*	*a b e f d c*	< 0.1 s
64	A97	R090	284	4	2	---	*a* (*b*) (*c*)	*d*	*a b*	*a b c d*	< 0.1 s
65	A98	R091	279	4	2	---	*a c*	*d*	*a c*	*a c b d*	< 0.1 s
66	A99	R092	275	4	2	---	*a b*	---	*a b*	*b a c d*	< 0.1 s

Table 10.3.1 (part 3 of 3): Schulze STV applied to instances of Tideman’s database

In 45 instances (A01, A02, A05, A08, A10, A12, A14, A18, A19, A20, A23, A48, A49, A51, A52, A54, A56, A57, A63, A64, A66, A68, A69, A70, A71, A72, A73, A74, A76, A78, A79, A80, A81, A83, A84, A85, A86, A87, A89, A92, A93, A94, A96, A98, A99), there are *M* Condorcet winners [according to (9.3.5)].

In 16 instances, there are *M*–1 Condorcet winners. For the remaining seat, there is a set $\emptyset \neq B \subsetneq A$ such that the Smith criterion for multi-winner elections (section 9.3) says that every winning set must contain at least one alternative from the set *B*. Table 10.3.2 lists these instances and the set *B*.

	name #1	*N*	*C*	*M*	Condorcet winners	set *B*
1	A03	989	15	7	*b d e f h k*	*g n*
2	A04	43	14	2	*i*	*a f k*
3	A06	280	9	5	*c e h i*	*b f g*
4	A07	79	17	2	*i*	*c d*
5	A13	104	26	2	*t*	*i k*
6	A15	77	21	2	*l*	*a b c d g h i j k m n p q r s t*
7	A17	867	13	8	*a b d e f j l*	*i m*
8	A22	44	11	2	*k*	*a b c d e g h i j*
9	A53	460	10	4	*a g j*	*d f*
10	A55	302	10	5	*a f i j*	*d e h*
11	A59	694	7	4	*d f g*	*b e*
12	A65	198	10	6	*b e f g j*	*a d*
13	A67	183	14	10	*b c e f g i j k l*	*h m n*
14	A88	135	9	6	*c e f g h*	*a b d*
15	A95	587	7	2	*a*	*b f*
16	A97	284	4	2	*a*	*b c*

Table 10.3.2: Instances of Tideman’s database with *M*–1 Condorcet winners

The 45 instances with M Condorcet winners are interesting because, in these instances, we know that we can remove all other alternatives or any subset of the other alternatives and the result will not change. This observation follows directly from the definition of Condorcet winners.

The 16 instances with M–1 Condorcet winners and a set B such that the last remaining winner must come from the set B are interesting because, again, we know that we can remove all other alternatives or any subset of the other alternatives and the result will not change. The proof for this is identical to the proof that the Schulze method satisfies Smith-IIA (section 4.7).

Only in 5 instances (A11, A33, A34, A35, A90), there are less than M–1 Condorcet winners. This shows that we succeeded in defining the Condorcet criterion for multi-winner elections in such a manner that there are always many Condorcet winners (section 9.3).

11. Comparison with other Methods

Table 11.2 compares the Schulze method with its main contenders. Extensive descriptions of the different methods can be found in publications by Fishburn (1977), Nurmi (1987), Kopfermann (1991), Levin and Nalebuff (1995), and Tideman (2006). As most of these methods only generate a set $\mathcal{S}$ of potential winners and don't generate a binary relation $\mathcal{O}$, only that part of the different criteria is considered that refers to the set $\mathcal{S}$ of potential winners.

In terms of satisfied and violated criteria, that election method, that comes closest to the Schulze method, is Tideman's ranked pairs method (Tideman, 1987). The only difference is that the ranked pairs method doesn't choose from the MinMax set $\mathcal{B}_D$.

The ranked pairs method works from the strongest to the weakest link. The link xy is locked if and only if it doesn't create a directed cycle with already locked links. Otherwise, this link is locked in its opposite direction.

In example 1, the ranked pairs method locks db. Then it locks cb. Then it locks ac. Then it locks ab, since locking ba in its original direction would create a directed cycle with the already locked links ac and cb. Then it locks cd. Then it locks ad, since locking da in its original direction would create a directed cycle with the already locked links ac and cd.

The winner of the ranked pairs method is alternative $a \notin \mathcal{B}_D = \{d\}$, because there is no locked link that ends in alternative a.

Although Tideman's ranked pairs method is that election method that comes closest to the Schulze method in terms of satisfied and violated criteria, random simulations by Wright (2009) showed that that election method, that agrees the most frequently with the Schulze method, is the Simpson-Kramer method (table 11.1).

number of alternatives	A	B	C
3	100.0 %	100.0 %	100.0 %
4	99.7 %	98.5 %	98.2 %
5	99.2 %	96.0 %	95.3 %
6	99.1 %	93.0 %	92.3 %
7	98.9 %	90.0 %	89.1 %

Table 11.1: Simulations by Wright (2009)

A: Probability that the Schulze method conforms with the Simpson-Kramer method

B: Probability that the Schulze method conforms with the ranked pairs method

C: Probability that the ranked pairs method conforms with the Simpson-Kramer method

	resolvability	Pareto	reversal symmetry	monotonicity	independence of clones	Smith	Smith-IIA	Condorcet	Condorcet loser	majority for solid coalitions	majority	majority loser	participation	MinMax set	prudence	polynomial runtime
Baldwin	Y	Y	N	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
Black	Y	Y	Y	Y	N	N	N	Y	Y	N	Y	Y	N	N	N	Y
Borda	Y	Y	Y	Y	N	N	N	N	Y	N	N	Y	Y	N	N	Y
Bucklin	Y	Y	N	Y	N	N	N	N	N	Y	Y	Y	N	N	N	Y
Copeland	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	Y
Dodgson	Y	Y	N	N	N	N	N	Y	N	N	Y	N	N	N	N	N
instant runoff	Y	Y	N	N	Y	N	N	N	Y	Y	Y	Y	N	N	N	Y
Kemeny-Young	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Nanson	Y	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
plurality	Y	Y	N	Y	N	N	N	N	N	N	Y	N	Y	N	N	Y
ranked pairs	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y
Schulze	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Simpson-Kramer	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	Y	Y
Slater	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Young	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	N	N

Table 11.2: Comparison of Election Methods

“Y” = compliance
“N” = violation

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